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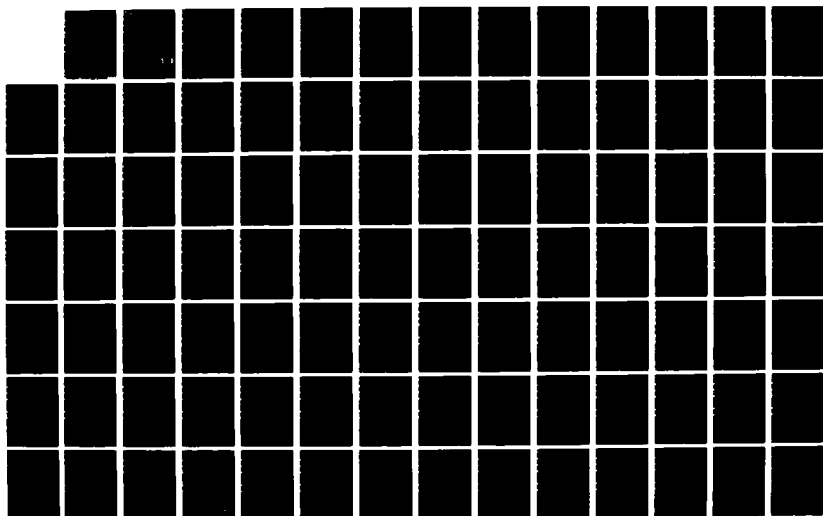
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(MAGNETO-PLASMA DYNAMIC) ENGIN (U) PURDUE UNIV
LAFAYETTE IN SCHOOL OF MECHANICAL ENGINEERING
R SHOURESHI ET AL 29 OCT 87

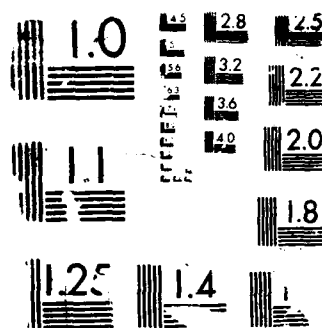
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DIAGNOSTICS FOR INTELLIGENT CONTROL OF MPD ENGINES

AFOSR-86-0278

SECOND ANNUAL REPORT

October 29, 1987

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EXECUTIVE SUMMARY

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Publications:

1. Murthy, S.N.B., Shoureshi, R., Pourki, F., "An approach to MPD engine instabilities," AIAA-87-0384, AIAA 25th Aerospace Science Meeting, January, 1987, Reno, Nevada.
2. Pourki, F., Goodfellow, K., Shoureshi, R., Murthy, S.N.B., "Observability of heat transfer to MPD thruster electrodes," AIAA-87-1070, 19th AIAA/DGLR/JSASS International Electric Propulsion Conference, May, 1987, Colorado Spring, Colorado.
3. Shoureshi, R., Pourki, F., "Controllability analysis of nonlinear distributed parameter systems," Paper #TA3-10:30, American Control Conference, June, 1987, Minneapolis, Minnesota.
4. Shoureshi, R., Pourki, F., "Stability analysis of electro-magneto plasma dynamics," under review for publication in the AIAA Journal of Guidance and Control.

Accomplishments: Two concurrent studies on stabilizability (Shoureshi, Pourki) and modeling (Murthy, Goodfellow) of MPD thrusters have been carried out. In the area of modeling the main accomplishments have been.

1. Formulation of a model for optimized nozzle geometry for a thermal-magnetic plasma dynamic thruster.
2. Formulation of a model for energy transfer process in the vicinity of electrodes in a thermal-magnetic plasma dynamic thruster.

Following is the highlights of the main accomplishments in stabilizability and observability of the MPD thrusters.

1. Lyapunov stability analysis of the equilibrium state of MPD thrusters for the general case of nonzero plasma velocity, and special case of zero velocity.
2. Stabilizability of the equilibrium state of MPD thrusters.
3. Derivation of the characteristics of a general, nonlinear hyperbolic system represented by partial differential equations (PDE), and their relations to controllability of the system.
4. Spectral analysis of the MPD thruster with zero plasma velocity.
5. Study of observability of nonlinear distributed parameter systems, and formulation of observability criteria for a special case of MPD thrusters.

MODEL OF PLASMA IN THE VICINITY OF AN ELECTRODE

PERSONNEL

S.N.B. Murthy, Principal Investigator
K. Goodfellow, Graduate Research Asst.

PUBLICATIONS DURING 1986-1987

1. Pourki, F., Goodfellow, K., Shoureshi, R., and Murthy, S.N.B., Observability of Heat Transfer to MPD Thruster Electrodes, International Electric Propulsion Conference, May 1987 AIAA-87-1070

2. Murthy, S.N.B., Shoureshi, R., and Pourki, F., An Approach to MPD Engine Instabilities, AIAA-87-0384.

3. Murthy, S.N.B., Plasmadynamic Thruster Nozzle, Accepted for 1987 JANAF Propulsion Meeting, Dec. 1987.

DISCUSSIONS HELD

1. Dr. A. Friedman, Dept. of Mathematics, Purdue University
2. Dr. A. Trippi, European Space Agency, Electric Propulsion Branch.
3. Dr. R. Hanson, Stanford University.
4. Dr. A. Eddy, Georgia Institute of Technology.
5. Dr. T.K. Bose, Indian Institute of Technology, Madras.
6. Dr. U. Daybelge, University of Istanbul, Turkey.

MAIN ACCOMPLISHMENTS

1. Formulation of a model for optimized nozzle geometry for a thermal-magnetic plasma dynamic thruster.

2. Formulation of a model for energy transfer processes in the vicinity of electrodes in a thermal-magnetic plasma dynamic thruster: Details on this model are provided in the following.

A MODEL FOR ELECTRODE VICINITY INTERACTIONS

Prepared by K. Goodfellow

1. INTRODUCTION

In a number of problems related to heat transfer to electrodes, it has been usual in the past to neglect the influence of plasma structure in the vicinity of the electrode in comparison with the overall thermal boundary layer. In some approaches to electrode surface deterioration, including wear, plasma or material jets and diffusional processes have been considered. However, most of those analyses also neglect the details of the plasma processes in the electrode region for which the characteristic lengths are Debye length, molecular mean free path, recombination length and Larmor radius. S.A. Self (1983,1987) and collaborators have included consideration of such near-electrode processes in the case of an isothermal plasma under certain other approximations.

We have utilized the S.A. Self approach as a bench mark and proceeded to set up a detailed model for electrode vicinity processes in the case of a thermal-magnetic thruster where the current flow and the electrode potential drop and charge are obviously of primary consideration.

The model is developed in the context of energy transfer to electrodes. However, one of the most important contributions of the model is to gain an understanding of the changes in those processes as a function of input current while the mass flux of heated gas is held constant. At some value of current, it is generally accepted, an instability, called onset instability, arises. The objective, then, is to establish the nature of possible measurements in the electrode vicinity that can provide an indication of the near-onset conditions in the thruster.

While we are yet to quantify the changes, it is our belief that in the case of a cathode, the model will demonstrate the importance of (a) the growth of ion number density, (b) the lag in heating of ions and (c) the gradual change in the electrode potential.

2. MODEL DESCRIPTION

A number of zones can be identified in the vicinity of the electrode based on characteristic length scales such as the recombination length (ℓ_R), the molecular mean free path ($\ell_{n,e,i}$ n, e and i referring to neutral, electron and ion, respectively) and the Debye length (ℓ_D). The rate-governed processes in those zones will depend further upon characteristic times. Figure 1 presents a schematic of the electrode region of a MPD engine. A thermal boundary layer is postulated that may be different from the momentum boundary layer. The Debye length and the mean free path of species are both small compared to the recombination length but ℓ_D may be either less than or be comparable to $\ell_{i,e}$.

The working fluid is assumed to be a monatomic gas that is undergoing single ionization so that the species considered are atoms, ions and electrons obeying Maxwell-Boltzmann statistics. The plasma in the free stream is assumed to be partially ionized with a possibility of both thermal and charge nonequilibrium. The electron and ion temperatures (T_e, T_i) are therefore assumed to be different. The Saha equation for reaction is modified in order to take into account non-equilibrium and inelastic collisions. The plasma is assumed to be subject to Joule heating and to induced and applied electro-magnetic fields.

The state of plasma within $\ell_{i,e}$ and therefore ℓ_D may only be described in terms of collisionless particles. It may be pointed out that $\ell_{i,e}$ and ℓ_D are not sharply defined boundaries.

Within the electrode, two regions have been identified in Figure 1: the first is a region where melting and evaporation may be taking place and the second, a region with pure conductive heat transfer. The first is referred to as a "mushy" region.

2.1. Model of Plasma in the Vicinity of an Electrode

A partially ionized two-temperature non-equilibrium plasma flow is considered, which obeys Maxwell-Boltzmann statistics. The induced magnetic field is included for consideration.

The plasma is assumed to be composed of three monatomic ideal gasses, electron, ion and neutral. The pressure of each gas species is given by $P_s = n_s k T_s$. The total gas pressure is then given by

$$P = \sum_s P_s \quad (2.1)$$

The energy of each species is the sum of the translational and the internal energies. The translational energy of each species is given by

$$e_s = \frac{3}{2} \frac{n_s k T_s}{m_s} \quad (2.2)$$

A Cartesian orthogonal coordinate system is used with coordinates that are parallel to the magnetic field, perpendicular to the magnetic field and perpendicular to the magnetic and the electric fields.

The electron current and heat flux are given by the following equations. The electron transport properties are significantly affected by the magnetic field and therefore these effects are included.

$$\vec{j}_e = \sigma_{\parallel} \epsilon_{\parallel} + \sigma_{\perp} \epsilon_{\perp} + \sigma_H \vec{b} \times \epsilon + \phi_{\parallel}^T \nabla_{\parallel} T_e + \phi_{\perp}^T \nabla_{\perp} T_e + \phi_H^T \vec{b} \times \nabla T_e \quad (2.3)$$

$$\vec{q}_e = -\frac{5}{2} \frac{kT_e}{e} \vec{j}_e - \lambda'_{\parallel} \nabla_{\parallel} T_e - \lambda'_{\perp} \nabla_{\perp} T_e - \lambda'_H \vec{b} \times \nabla T_e - T_e \phi_{\parallel}^T \epsilon_{\parallel} - T_e \phi_{\perp}^T \epsilon_{\perp} - T_e \phi_H^T \vec{b} \times \epsilon \quad (2.4)$$

The transport coefficients (σ , ϕ^T , and λ') are presented, for example in Mitchner and Kruger[1974], in the form of integral functions of C . The transport coefficients are also available in other forms from sources such as Bose [1987].

The heavy particle transport properties are not affected by the presence of a magnetic field unless the field is very strong. For the system considered the magnetic field is assumed to be sufficiently weak so that the heavy particle transport equations can be written for a partially ionized plasma without a magnetic field. Accordingly, the following relations may be written for the heavy particle transport properties.

$$\tau_{\alpha\beta} = \eta \left\{ \frac{\partial u_{\alpha}}{\partial x_{\beta}} + \frac{\partial u_{\beta}}{\partial x_{\alpha}} - \frac{2}{3} (\nabla \cdot \vec{u}) \delta_{\alpha\beta} \right\} \quad (2.5)$$

$$\vec{q}_h = \frac{5}{2} kT_h \sum_s n_s \vec{U}_s - \lambda' \nabla T_h - n_h kT_h \sum_s \frac{D_s^T}{\rho_s} \vec{d}_s \quad (2.6)$$

$$\vec{j}_h = \vec{j}_i = en_i \vec{U}_i \quad (2.7)$$

$$\vec{U}_s = \frac{n^2}{n_s \rho} \sum_k m_k D_{sk} \vec{d}_k - \frac{1}{\rho_s} D_s^T \nabla (\ln T_h) \quad (2.8)$$

$$\vec{d}_s \equiv \nabla \left(\frac{n_s}{n} \right) + \left(\frac{n_s}{n} - \frac{\rho_s}{\rho} \right) \nabla (\ln P) - \frac{\rho_s}{P \rho} \left(\frac{\rho}{m_s} \vec{F}_s - \sum_k n_k \vec{F}_k \right) \quad (2.9)$$

where

$\delta_{\alpha\beta}$ = Kronecker delta = 0 if $\alpha \neq \beta$, 1 if $\alpha = \beta$

D_{sk} = concentration diffusion coefficient

D_s^T = thermal diffusion coefficient

Since a non-equilibrium plasma is considered, a relationship for the generation of species is required. Only collisional reactions will be considered. Specifically the three-body recombination reaction where the third body is an electron may be written as follows.



The generation rate equation is then given by the following.

$$\dot{n}_e = \frac{\partial n_e}{\partial t} = \alpha(T_e) \left\{ (n_e n_i)^* - n_e n_i \right\} \quad (2.11)$$

where

$$(n_e n_i)^* \equiv n_n \left\{ \frac{n_e n_i}{n_n} \right\}_{\text{equil}} \quad (2.12)$$

$$\alpha(T_e) = n_e 1.09 \times 10^{-20} T_e^{-9/2} \frac{m^3}{s} \quad (2.13)$$

= Hinnov-Hirschberg recombination coefficient

The equilibrium concentrations are given by the Saha equation, namely.

$$\left\{ \frac{n_e n_i}{n_n} \right\}_{\text{equil}} = 2 \frac{g_i}{g_n} \left\{ \frac{2\pi m_e k T_e}{h^2} \right\}^{\frac{3}{2}} \exp \left\{ \frac{-\epsilon_i}{k T_e} \right\} \quad (2.14)$$

where g_s is equal to the electron energy partition function and ϵ_i , the ionizational energy. Since a singly ionized gas is considered, the electron and ion generation rates are equal ($\dot{n}_e = \dot{n}_i$).

2.2. Energy Balance in the Vicinity of a Cathode

As a particular example of electrode-associated processes, we examine the energy balance in the vicinity of a cathode. The equations describing the state of the plasma including non-equilibrium and multi-temperature effects are presented. Referring to Figure 1, the energy balance equations applicable to different regions are presented along with the associated current flux equation. It is assumed that the cathode material is catalytic and recombination processes within the material give rise to a flux of neutrals into the Debye region and also to heat generation within the electrode. A region, referred to as the "mushy" region, is identified in the electrode material adjoining the plasma. In that region it is assumed that phase change processes, solid to liquid, liquid to vapor and also solid to vapor, are expected to occur.

The energy equations are written specifically for each region shown in Figure 1. In all regions the plasma is assumed to be non-neutral ($n_e \neq n_i$) and two-temperature governed ($T_e \neq T_i$).

2.2.1. Region 1 (solid)

The energy equation in the solid material is given by the well-known Stephan or heat equation, namely.

$$\dot{E}_{st} = (\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} \quad (2.15a)$$

or

$$\rho_{\text{solid}} C_{p_{\text{solid}}} \frac{\partial T}{\partial t} = \nabla \left(\lambda_{\text{solid}} \nabla T \right) + \frac{j^2}{\sigma_{\text{solid}}} \quad (2.15b)$$

where \dot{E}_{st} is the energy stored in the material, $(\dot{E}_{in} - \dot{E}_{out})$, net energy transfer through the material by conduction, and \dot{E}_{gen} , heat generated within the material due to Joule heating.

The current flux is given by

$$\vec{j} = \sigma_{\text{solid}} \vec{E} \quad (2.16)$$

The energy input to the solid region is determined from an energy balance at the surface $y = 0$. The energy balance yields namely,

$$\left[\lambda_{\text{solid}} \frac{\partial T_{\text{solid}}}{\partial y} \right]_{y=0} = \left[\lambda_{\text{mushy}} \frac{\partial T_{\text{mushy}}}{\partial y} \right]_{y=0} + H_m \frac{\partial y}{\partial t} \quad (2.17)$$

where H_m is the latent heat of fusion of the material

2.2.2. Region 2 (Mushy)

Region 2 is the "mushy" region where solid, liquid and gaseous states may be present simultaneously. The energy for the "mushy" region is similar to the energy equation for the solid region except that an additional term is needed to account for the energy associated with the material phase change, $\dot{Q}^*(T)$. If the material is assumed to "pure" then there will be no conduction across the mushy region. The region will be at a uniform temperature because it is undergoing a phase change. If the material is not considered "pure" then there can be conduction across the region because there may be a small temperature gradient. The energy transfer in the mushy region is then given by the following model equation [Fasano and Primicerio 1981], which is in the nature of a modified Stephan equation.

$$\rho_{\text{mushy}} C_{p_{\text{mushy}}} \frac{\partial T}{\partial t} = \nabla \left(\lambda_{\text{mushy}} \nabla T \right) + \frac{j^2}{\sigma_{\text{mushy}}} + \dot{Q}^*(T) \quad (2.18)$$

where

$$\vec{j} = \sigma_{\text{mushy}} \vec{E} \quad (2.19)$$

and $\sigma_{\text{mushy}} = \sigma_{\text{mushy}}(T)$ = electrical conductivity of the mushy region which is temperature dependent.

The two surfaces of the mushy region at $y = 0$ and $y = y_1$, are not fixed. The surface at $y = 0$ is allowed to recess into the solid material as melting occurs. Melting will occur in the region as long as the energy input to the region (at the surface $y = y_1$ and from Joule heating) exceeds the energy used in the phase transition plus the energy removed to the solid region by conduction. If the energy inputs and outputs are equal the boundary will not recede. The surface at $y = y_1$ will recede because of the evaporation of material at the surface.

The energy input to the mushy region from the plasma is determined from an energy balance evaluated at the surface $y = y_1$, as shown in Figure 1. The cathode is assumed to have a catalytic surface where incident ions and electrons recombine and are re-emitted as neutrals. Thermionic emission may also occur if the cathode

temperature is sufficiently high. It should be noted that the thermionically emitted electrons provide an additional localized current which in turn produces an additional localized Joule heating. The localized heating may lead to localized evaporation or to the eruption of material. The overall energy balance equation is then as follows.

$$\dot{Q}^* + \dot{q}_{\text{cond}} = \dot{q}_p + \dot{q}_{\text{rad}} + \dot{Q}_{\text{sur}} \quad (2.20)$$

where \dot{q}_p is the net energy transfer to the surface from the plasma due to the kinetic energy and potential energy of the particles, \dot{q}_{rad} , the net radiation to the surface from the plasma, \dot{Q}_{sur} , the energy generation at the surface due to recombination of ions and electrons \dot{Q}^* , the energy associated with phase change and, \dot{q}_{cond} , energy conducted into the mushy region.

2.2.3. Region 3 (Sheath)

The region immediately adjacent to the electrode and within a distance of the order of Debye length (ℓ_D) of the surface is the sheath region. In this region, charge separation occurs and a net negative charge exists because of an excess number of electrons. The region is considered collisionless in the sense that only electron-neutral and ion-neutral collisions are present. These collisions are included because of the large number of neutrals in the region. Since neutrals being emitted from the surface do not experience a force from the fields, they tend to remain near the surface. They are removed through diffusion driven by the concentration gradient. All other collisions are assumed to be negligible.

The energy transfer to the surface from the plasma is from the collision of particles on the surface. Since a cathode is being considered the particles of interest are the ions. The energy of the particles is in two forms, the kinetic energy due to their motion and the potential energy associated with moving charged particles through an electric potential. The electric potential will tend to move the electrons away from the electrode while accelerating the ions towards the electrode.

Since the interest is in particles that strike the surface, the velocity component normal to the surface is the one of interest. The describing energy equations are given by the following. [Mitchner and Kruger 1974]

The species heat-flux vector, \vec{q}_s , is

$$\vec{q}_s = \int_{-\infty}^{\infty} \frac{1}{2} n_s m_s C^2 C_{\perp} f_s d^3c \quad (2.21)$$

and the particle potential energy, W_s , is

$$W_s = \int_{y=y_1}^y \zeta_s \phi_s dy \quad (2.22)$$

where the electric potential ϕ_s is given by the following Poisson's equation.

$$\nabla^2 \phi_s = \int_{-\infty}^{\infty} \zeta_s f_s d^3c \quad (2.23)$$

The species current-flux vector, \vec{J}_s , given by

$$\vec{J}_s = \int_{-\infty}^{\infty} n_s \zeta_s C_{\perp} C_{\perp} f_s d^3c \quad (2.24)$$

The distribution f_s is determined from the Boltzmann equation, namely

$$\begin{aligned} \frac{\partial}{\partial t} (n_s f_s) + (u_\beta + C_\beta) \frac{\partial}{\partial x_\beta} (n_s f_s) + \left\{ \frac{F_{s\beta}}{m_s} - \frac{Du_\beta}{Dt} \right\} n_s \frac{\partial f_s}{\partial C_\beta} \\ - C_\beta n_s \frac{\partial f_s}{\partial C_\gamma} \frac{\partial u_\gamma}{\partial x_\beta} = \sum_k \Psi_{sk} \end{aligned} \quad (2.25)$$

where Ψ_{sk} is the rate of increase of the property of interest (mass, momentum, energy or charge) due to collisions between particles of type s with particles of type k.

2.2.4. Region 4

Region 4 is similar to region 3, as they are both considered collisionless. The describing equations are therefore the same for the two regions. Since this region is outside of the sheath, it is expected to contain a greater number of ions than region 3. The electron-neutral and ion-neutral collisions also become negligible because of a large decrease in the number of neutrals in the region. The collision parameter Ψ_{sk} in equation 2.25 is therefore equal to zero.

2.2.5. Regions 5 and 6

Regions 5 and 6 are assumed to be collision-dominated and can therefore be described using the hydrodynamic approximation. Particle recombination is assumed to be absent in Region 5 because it is within a distance of the order of the recombination length ℓ_R of the electrode surface. The recombination is included in Region 6. The resulting energy balance equations for Region 5 are as follows.

$$\begin{aligned} \frac{D}{Dt} \left(\frac{3}{2} n_e k T_e \right) + \left(\frac{5}{2} n_e k T_e \right) \nabla \cdot \vec{u}_e = -\nabla \cdot \vec{q}_e + \tau_e \nabla \cdot \vec{u}_e + \vec{j}_e \cdot \vec{E}'_e \\ - \frac{2m_e}{m_h} \bar{\nu}_{eh} n_e \frac{3}{2} k (T_e - T_h) - \dot{R}_e \end{aligned} \quad (2.26)$$

$$\begin{aligned} \frac{D}{Dt} \left(\frac{3}{2} n_h k T_h \right) + \frac{5}{2} n_h k T_h \nabla \cdot \vec{u}_h = -\nabla \cdot \vec{q}_h + \tau_h \nabla \cdot \vec{u}_h + \vec{j}_i \cdot \vec{E}'_i \\ - \frac{2m_e}{m_h} \bar{\nu}_{he} n_h \frac{3}{2} k (T_h - T_e) - \dot{R}_h \end{aligned} \quad (2.27)$$

where

$$\tau \nabla \cdot \vec{u} = \tau_{\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta}; \quad n_h = n_i + n_n; \quad m_n \approx m_i \approx m_h$$

The energy equations for Region 6 are given by the following. The recombination energy is included in the electron energy equation.

$$\begin{aligned} \frac{D}{Dt} \left(n_e \left(\frac{3}{2} kT_e + \epsilon_i \right) \right) + n_e \left(\frac{5}{2} kT_e + \epsilon_i \right) \nabla \cdot \vec{u}_e = -\nabla \cdot \left(\vec{q}_e - \frac{\epsilon_i}{e} \vec{j}_e \right) \\ + \tau_e \nabla \vec{u}_e + \vec{j}_e \cdot \vec{E}'_e - \frac{2m_e}{m_h} \bar{\nu}_{eh} n_e \frac{3}{2} k (T_e - T_h) \\ - \dot{R}_e \end{aligned} \quad (2.28)$$

$$\begin{aligned} \frac{D}{Dt} \left(\frac{3}{2} n_h kT_h \right) + \frac{5}{2} n_h kT_h \nabla \cdot \vec{u}_h = -\nabla \cdot \vec{q}_h + \tau_h \nabla \vec{u}_h + \vec{j}_i \cdot \vec{E}'_i \\ - \frac{2m_e}{m_h} \bar{\nu}_{he} n_h \frac{3}{2} k (T_h - T_e) - \dot{R}_h \end{aligned} \quad (2.29)$$

2.2.7. Region 7 (Free Stream)

In the free stream region, the various gradients are assumed to very small compared to the other terms and are neglected. The rate of change of energy within the system is equal to the generated energy from Joule heating minus the energy lost from radiation, recombination and collisions. The energy equations can therefore be simplified as follows.

$$\frac{\partial}{\partial t} \left(n_e \left(\frac{3}{2} kT_e + \epsilon_i \right) \right) = \vec{j}_e \cdot \vec{E}'_e - \frac{2m_e}{m_h} \bar{\nu}_{eh} n_e \frac{3}{2} k (T_e - T_h) - \dot{R}_e \quad (2.30)$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} n_h kT_h \right) = \vec{j}_i \cdot \vec{E}'_i - \frac{2m_e}{m_h} \bar{\nu}_{he} n_h \frac{3}{2} k (T_h - T_e) - \dot{R}_h \quad (2.31)$$

$$\vec{j}_e = \sigma_{||} \epsilon_{||} + \sigma_{\perp} \epsilon_{\perp} + \sigma_H \vec{b} \times \epsilon \quad (2.32)$$

3. PROPOSED SCHEME FOR UTILIZATION OF THE MODEL

The vicinity of an electrode, in particular the cathode, in an MPD thruster is considered according to the simplified model presented in Figure 2. The geometry and the flow rate of propellant are considered fixed. The thruster is then expected to be operated at two current levels, namely a low current level in which the plasma is partially ionized and a high current level at which the plasma is nearly fully ionized.

The objective is to establish a methodology for determining the plasma properties in the cathode vicinity and for relating changes in such properties with respect to current to the occurrence of onset conditions.

The describing equations for the model are considered in two forms, a collision-dominated plasma and a collisionless plasma. The sheath and free-fall zones are considered as collisionless, since they fall within a distance less than one mean free path ($\ell_{i,e}$) from the electrode surface. Plasma that is located at $y > \ell_{i,e}$ is considered as collision dominated plasma. For the purposes here Regions 3, 4 and 5 from Figure 1 are considered, that is the plasma within the recombination length of the cathode surface.

Assumptions:

1. One dimensional model (normal to the cathode surface).
2. Maxwell-Boltzmann statistics apply to the particles.
3. Isotropic properties.
4. Negligible convection.
5. Negligible magnetic field (induced and/or applied).
6. Cathode absorbs all incident species (ions are reemitted as neutrals).
7. Monoenergetic particles (ions and thermionic electrons).
8. Thermionic electrons are emitted at the cathode temperature.
9. No recombination or ionization occurs in $y < \ell_R$.
10. The energy of the thermionic (beam) electrons is negligibly small in the collision dominated region.
11. Collisions are included for the region $\ell_{i,e} < y < \ell_{e,i}$.
12. Diffusion neglected in the $y < \ell_{i,e}$.

3.1. Governing Equations

3.1.1. Collisionless Region

The sheath and free-fall regions are considered to contain a collisionless plasma, that is, no interparticle collisions occur within these regions. The total energy of each particle is therefore constant since energy cannot be gained or lost by a particle without a collision. Three species of particles are considered, positive ions, plasma electrons and thermionic (or beam) electrons. Neutral particles will also be present.

The solution procedure is to solve for the number density of each species using the energy and continuity equations and then use Poissons equation to solve for the electric field (E) and the electric potential (ϕ). Once the electric potential is determined, the number densities and and velocities of each species can be determined.

Electrons produced by thermionic emission at the cathode surface are accelerated through the region by the electric field. Their total energy is composed of two parts, the particle kinetic energy and the potential energy in the form:

$$E_b = \frac{1}{2} m_b v_b^2 - e\phi = \text{constant} \quad (3.1)$$

The beam electrons are assumed to be emitted with temperatures equal to the cathode temperature (T_c). The initial beam electron energy is therefore:

$$E_c = \frac{1}{2} m_b v_{bo}^2 = \frac{3}{2} k T_w \quad (3.2)$$

Using equations 3.1 and 3.2, the beam electron energy can be written in the form:

$$\frac{1}{2} m_b v_b^2 = e(\phi_c - \phi) + E_c, \quad (3.3)$$

which shows that the kinetic energy is equal to the change in potential energy plus the initial kinetic energy. The initial kinetic energy is expected to be very small compared to the other terms but is retained to prevent a possible singularity from occurring at the electrode surface ($\phi = \phi_c$). The potentials are used as absolute values ($\phi = |\phi|$, $\phi_c = |\phi_c|$). The flux of beam electrons within the region is continuous, since no ionization or recombination occurs. The continuity equation yields:

$$j_b = en_b v_b \quad (3.4)$$

Solving equations 3.3 and 3.4 for the number density, n_b yields:

$$n_b = \frac{j_b}{e} \left\{ \frac{m_b}{2} \right\}^{1/2} \left\{ e(\phi_c - \phi) + E_c \right\}^{-1/2} \quad (3.5)$$

The thermionic emission current is assumed to be dependent only on the cathode temperature and material and is given by the empirical relationship:

$$j_b = (\rho A) T^2 10^{-\frac{5040 \psi}{T}} \quad (3.6)$$

where ρA and ψ are material properties and T is the electrode temperature in degrees Kelvin [Smithells 1952]. For the purposes here, the effect of space charge on thermionic emission has been neglected, that is, a space charge in the vicinity of the cathode acting either to enhance or to inhibit the electron emission is neglected.

The ions are assumed to be monoenergetic with an initial kinetic energy of $\frac{1}{2} m_i v_{io}^2$ at the plane where they enter the collisionless region ($y = \ell_{i,e}$). The ion energy equation can be written as:

$$\frac{1}{2} m_i v_i^2 - e\phi = E_{io} \quad (3.7)$$

where E_{io} is the initial ion energy at $y = \ell_{i,e}$

$$E_{io} = \frac{1}{2} m_i v_{io}^2 - e\phi_2 \quad (3.8)$$

or

$$E_{io} = e\phi_o \quad (3.9)$$

where ϕ_o is the equivalent potential drop for the ion to obtain the energy E_{io} . From equations 3.7 and 3.9 the ion velocity can be determined as:

$$v_i = \left(\frac{2e}{m_i} \right)^{1/2} \left(\phi_o + \phi \right)^{1/2} \quad (3.10)$$

The ion distribution is continuous due to the absence of collisions and continuity yields:

$$j_i = en_i v_i = en_{i0} v_{i0} \quad (3.11)$$

where n_{i0} and v_{i0} represent the ion number density and velocity at the plane where the ions enter the collisionless region at $y = \ell_{i,e}$. Solving for n_i yields:

$$n_i = \frac{j_i}{e} \left\{ \frac{m_i}{2e} \right\}^{1/2} \left\{ \phi_o + \phi \right\}^{-1/2} \quad (3.12)$$

The plasma electrons are assumed to be Maxwellian particles. The number density is given by:

$$n_e = n_{e2} \exp \left\{ \frac{-e\phi + E_{e0}}{kT_e} \right\} \quad (3.13)$$

where E_{e0} is the electron kinetic energy at the free-fall edge $y = \ell_{i,e}$, the potential energy is $e\phi_2$, and the number density is n_{e2} . The energies are related such that:

$$E_{e0} - e\phi_2 = 0 \quad (3.14)$$

Equation 3.13 can be rewritten as:

$$n_e = n_{e2} \exp \left[\frac{-e(\phi - \phi_2)}{kT_e} \right] \quad (3.15)$$

The electron current in the collisionless region is considered in two parts. The electrons that move towards the cathode with initial energies, $E_{e0} < e(\phi_c - \phi_2)$ are repelled and return to the collision dominated region. The electrons with $E_{e0} \geq e(\phi_c - \phi_2)$ cross the collisionless region and strike the cathode. These high energy electrons make up the electron current j_e .

$$j_e = \frac{n_{e2}}{4} \left\{ \frac{8kT_e}{\pi m_e} \right\}^{1/2} \exp \left\{ \frac{-e(\phi_c - \phi_2)}{kT_e} \right\} \quad (3.16)$$

The solution of Poisson's equation in the sheath region will be different from the solution in the free-fall region because of charge separation within the sheath. Poisson's equation is used to solve for the electric potential distribution (ϕ). Substituting the particle number densities into Poisson's equation yields:

$$\frac{d^2\phi}{dy^2} = \rho^c = \frac{e}{\epsilon_0} (n_i - n_e - n_b) \quad (3.17)$$

or

$$\begin{aligned} \frac{d^2\phi}{dy^2} = \frac{e}{\epsilon_0} & \left[\frac{j_i}{e} \left\{ \frac{m_i}{2e} \right\}^{\frac{1}{2}} \left\{ \phi_0 + \phi \right\}^{-\frac{1}{2}} - \frac{j_b}{e} \left\{ \frac{m_b}{2} \right\}^{\frac{1}{2}} \left\{ e(\phi_c - \phi) + E_c \right\}^{-\frac{1}{2}} \right. \\ & \left. - n_{eo} \exp \left(\frac{-e(\phi - \phi_2)}{kT_e} \right) \right] \quad (3.18) \end{aligned}$$

Since j_i , j_b and the guessed value of ϕ_c are known constants, equation 3.18 can be integrated once using the integrating factor $\frac{d\phi}{dy}$ and the given boundary conditions $\phi = \phi_3$ and $E = E_3$ at $y = \ell_D$ to yield:

$$\begin{aligned} \frac{1}{2} (E^2 - E_3^2) = & 2J_i \left\{ \left(\phi_0 + \phi \right)^{\frac{1}{2}} - \left(\phi_0 + \phi_3 \right)^{\frac{1}{2}} \right\} \\ & + \frac{2J_b}{e} \left\{ \left(e(\phi_c - \phi) + E_c \right)^{\frac{1}{2}} - \left(e(\phi_c - \phi_3) + E_c \right)^{\frac{1}{2}} \right\} \\ & + N_{eo} \frac{kT_e}{e} \left\{ \exp \left(\frac{-e\phi}{kT_e} \right) - \exp \left(\frac{-e\phi_3}{kT_e} \right) \right\} \quad (3.19) \end{aligned}$$

Poisson's equation can be used again to write equation 3.19 in the form:

$$\begin{aligned} E = \frac{d\phi}{dy} = & \left[4J_i \left\{ \left(\phi_0 + \phi \right)^{\frac{1}{2}} - \left(\phi_0 + \phi_3 \right)^{\frac{1}{2}} \right\} \right. \\ & + \frac{4J_b}{e} \left\{ \left(e(\phi_c - \phi) + E_c \right)^{\frac{1}{2}} - \left(e(\phi_c - \phi_3) + E_c \right)^{\frac{1}{2}} \right\} \\ & \left. + 2N_{eo} \frac{kT_e}{e} \left\{ \exp \left(\frac{-e\phi}{kT_e} \right) - \exp \left(\frac{-e\phi_3}{kT_e} \right) \right\} + E_3^2 \right]^{\frac{1}{2}} \quad (3.20) \end{aligned}$$

$$J_i = \frac{j_i}{\epsilon_0} \left\{ \frac{m_i}{2e} \right\}^{1/2} \quad (3.21)$$

$$J_b = \frac{j_b}{\epsilon_0} \left\{ \frac{m_b}{2} \right\}^{1/2} \quad (3.22)$$

$$N_{eo} = \frac{n_{eo} e}{\epsilon_0} \exp \left\{ \frac{e \phi_2}{kT_e} \right\} \quad (3.23)$$

Equation 3.20 can then be numerically integrated to determine the electric potential distribution within the sheath. Once the potential is determined, the number densities of each species, the electric field and the space charge can be determined using equations 3.5, 3.12, 3.13, and 3.17 respectfully.

A separate equation is needed to determine if the guessed value of ϕ_c is correct. This equation is obtained by performing a force balance on the ions. This yields equation 3.24 which has a resemblance to Ohms law for a plasma with collisions.

$$E = \frac{-m_i}{e^3} \frac{j_i}{n_i^2} \left\{ \frac{dj_i}{dy} - \frac{j_i}{n_i} \frac{dn_i}{dy} \right\} \quad (3.24)$$

which can be simplified for the case at hand where $j_i = \text{constant}$, to yield

$$E = \frac{m_i}{e^3} \frac{j_i^2}{n_i^3} \frac{dn_i}{dy} \quad (3.25)$$

The derivation of equation is given in appendix I. A new value for ϕ_c is obtained by applying equations 3.20 and 3.25 at the cathode surface. This procedure is repeated, starting again from $y = \ell_{i,e}$, until the guessed and calculated values of ϕ_c are within a chosen tolerance.

The solution of Poisson's equation is simplified within the free-fall region due to the absence of charge separation, that is ρ^c being constant. The value of ρ^c is determined by the charge at the edge boundary ($y = \ell_{i,e}$). Poisson's equation can be integrated once to yield:

$$-\frac{d\phi}{dy} = E = E_2 - \frac{\rho^c}{\epsilon_0} (\ell_{i,e} - y) \quad (3.26)$$

and integrated again to yield:

$$\begin{aligned} \phi &= E_2 (\ell_{i,e} - y) \\ &+ \frac{\rho^c}{\epsilon_0} \left\{ \frac{\ell_{i,e}^2 - y^2}{2} - \ell_{i,e} (\ell_{i,e} - y) \right\} + \phi_2 \end{aligned} \quad (3.26)$$

3.1.2. Collision Dominated Region

The governing equations for the collision dominated region are given by the hydrodynamic plasma equations. These equations are applicable when the number of collisions within the plasma are sufficiently large so that average properties can be assigned to the plasma properties. The equations are further simplified here for the one dimensional case. The derivation of these equations can be found in a number of references such as Mitchner and Kruger[1974].

Charge density

$$\rho^c = e \left(n_i - n_e - n_b \right) \quad (3.28)$$

Current density

$$j = j_i + j_b - j_e \quad (3.29)$$

$$j_e = en_i U_i \quad (3.30)$$

$$j_e = en_e U_e \quad (3.31)$$

$$j_b = en_b U_b \quad (3.32)$$

Poisson's equation

$$\frac{d^2 \phi}{dy^2} = -\frac{dE}{dy} = \frac{\rho^c}{\epsilon_0} \quad (3.33)$$

State

$$P = k \left(n_e T_e + n_b T_b + (n_i + n_n) T_h \right) \quad (3.34)$$

Ohm's law (first approximation)

$$j = \sigma E \quad (3.35)$$

Energy

$$\frac{d}{dy} \left\{ \lambda_e \frac{dT_e}{dy} \right\} = j_e E - \sum_h \frac{2m_e}{m_h} \bar{\nu}_{eh} n_e \frac{3}{2} k (T_e - T_h) \quad (3.36)$$

$$\frac{d}{dy} \left\{ \lambda_h \frac{dT_h}{dy} \right\} = j_i E - \frac{2m_e}{m_h} \bar{\nu}_{he} (n_i + n_n) \frac{3}{2} k (T_h - T_e) \quad (3.37)$$

Diffusion

$$U_e = -\mu_e \left\{ E + \frac{k}{en_e} \left(T_e \frac{dn_e}{dy} + n_e \frac{dT_e}{dy} \right) \right\} \quad (3.38)$$

$$U_i = \mu_i \left\{ E - \frac{k}{en_i} \left(T_h \frac{dn_i}{dy} + n_i \frac{dT_h}{dy} \right) \right\} \quad (3.39)$$

$$U_n = -\mu_n \left\{ \frac{k}{en_n} \left(T_h \frac{dn_n}{dy} + n_n \frac{dT_h}{dy} \right) \right\} \quad (3.40)$$

$$U_b = -\mu_b \left\{ E + \frac{k}{en_b} \left(T_b \frac{dn_b}{dy} \right) \right\} \quad (3.41)$$

These equations are used to solve for the variables n_e , n_i , n_n , n_b , T_e , T_h , j_e , j_i , ρ^c , U_i , U_e , U_n , U_b , E and ϕ .

The characteristic length scales are given by the equations:

$$\ell_{i,e} = \frac{1}{n_e Q_{ie}} \quad (3.42)$$

where Q_{ie} is the collisional cross-section. The sheath thickness is determined using the Child-Langmuir law [Chen 1965].

$$\ell_D^2 = \left\{ \frac{2}{m_e e} \right\}^{\frac{1}{2}} \frac{1}{9\pi} \frac{(\phi_c - \phi)^{\frac{3}{2}}}{j} \left\{ 1 + \frac{2.66}{\sqrt{\eta}} \right\} \quad (3.43)$$

where

$$\eta = \frac{e(\phi_c - \phi)}{kT_e} \quad (3.44)$$

Alternatively, or the following equation can be used.

$$\ell_D = \left[\frac{\epsilon_0 kT_e}{n_{e2} e^2} \right]^{\frac{1}{2}} \quad (3.45)$$

The sheath thickness is equal to the larger of the two values for ℓ_D from equations 3.43 and 3.45.

4. PLANNED SOLUTION PROCEDURE

The overall solution procedure is outlined in the flow chart in Figure 3. The general solution is found by using the solutions for each of the regions. The governing equations are solved for each region starting with the collision-dominated region and proceeding towards the cathode surface. The parameter values at the boundaries are determined by the preceding region and are used as the beginning boundary conditions for the next region. The process is repeated until the estimated values for collisionless region potential drop and beam electron number density agree with the calculated values. Once convergence is achieved, the distributions of the various parameters can be studied.

Four test cases are expected to be examined. They are for combinations of high and low current density levels and high and low cathode temperatures. It is expected that estimation of distributions of the critical parameters can be determined by examining the results of these four cases.

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NOMENCLATURE

Symbol	Description
\vec{B}	magnetic field
\vec{c}	particle velocity in laboratory reference frame
\vec{C}	peculiar particle velocity
\vec{E}	applied electric field
\vec{F}	body force on particle
\vec{j}	current flux
\vec{q}	heat flux
\vec{u}	mass velocity
\vec{U}	diffusion velocity in fluid reference frame
C_p	specific heat
e	electron charge
E	energy
E	electric field (1-D)
h	Planck constant
j	current density (1-D)
k	Boltzmann constant
m	mass
M	gas molecular weight
n	number density
P	pressure
Q^*	phase change energy
R	radiation energy
T	temperature

t	time
v	velocity (1-D)
W	potential energy
x	position
\vec{E}	general electric field
ϵ_i	ionization potential
ϵ_0	permittivity constant
η	viscosity
λ, λ'	thermal conductivity
ϕ	electric potential
ϕ^T	thermal diffusion
ρ	density
σ	electrical conductivity
μ	mobility
$\bar{\nu}_{sk}$	mean collision frequency of type s particles with type k particles
τ	shear stress
q_s	particle charge
ℓ_D	Debye length
ℓ	mean free path

Subscripts

b	thermionic (beam) electrons
c	cathode
cond	conduction
e	electron
i	ion

k	species of type k
mushy	mushy region
n	neutral
o	reference state
s	species of type s
rad	radiation
solid	solid material region
sur	surface
	parallel to magnetic field
⊥	perpendicular to magnetic field
H	mutually perpendicular to magnetic and electric fields
α, β, γ	directions
∞	free stream value
1	boundary $y = \ell_R$
2	boundary $y = \ell_{i,e}$
3	boundary $y = \ell_D$

Definitions

$$\vec{b} = \frac{\vec{B}}{B}$$

$$\bar{C}_s = \text{mean particle speed} = \left\{ \frac{8kT_s}{\pi m_s} \right\}^{1/2}$$

$$\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$$

$$\vec{\tau} = \vec{E}' + \frac{\Delta P_e}{n_e e}$$

f_s = velocity distribution function

$\rho_s = n_s m_s$ = species density

$$\rho = \sum_s \rho_s$$

$$m_{sk} = \text{reduced mass} = \frac{m_s m_k}{m_s + m_k}$$

$$\dot{n}_s = \text{generation rate of species } s$$

$$\ell_R = \text{recombination length} = \left\{ \frac{2D_a}{\beta_{n\infty}^2} \right\}^{1/2}$$

$$D_a = \text{ambipolar diffusion coefficient} = \frac{\mu_e D_i + \mu_i D_e}{\mu_e + \mu_i}$$

$$D_i = \text{ion diffusion coefficient} = \frac{kT_h}{e} \mu_i$$

$$D_e = \text{electron diffusion coefficient} = \frac{kT_e}{e} \mu_e$$

$$\mu_e = \text{electron mobility} = \frac{e}{m_e \bar{\nu}_{eH}}$$

$$\mu_i = \text{ion mobility} = \frac{\rho_n}{\rho} \frac{\bar{\nu}_{en}}{\bar{\nu}_{eH}} + \frac{\rho_i}{\rho} \mu_{ie}$$

$$\mu_{in} = \frac{e}{m_{in} \bar{\nu}_{in}}$$

$$\mu_{ie} = \frac{e}{m_i \bar{\nu}_{eH}}$$

$$\vec{u}_s = \vec{u} + \vec{U}_s$$

$$\vec{u} = \frac{1}{\rho} \sum_s \rho_s \vec{U}_s$$

$$\ell = \text{mean free path} = \frac{\bar{c}}{\nu}$$

$$\frac{D(\cdot)}{Dt} \equiv \frac{\partial(\cdot)}{\partial t} + \vec{u} \cdot \nabla(\cdot)$$

APPENDIX I

A relationship between current and electric field for a collisionless region can be developed as follows. The total current is given by

$$j = j_i + j_b - j_e \quad (I.1)$$

where the electron and beam currents are constant for a given cathode temperature and collisionless region electric potential drop.

$$j_b = (\rho A) T^2 10^{-\frac{5040 \psi}{T}} \quad (I.2)$$

$$j_e = \frac{n_{eo}}{4} \left\{ \frac{8kT_e}{\pi m_e} \right\}^{\frac{1}{2}} \exp \left\{ \frac{-e(\phi_c - \phi_2)}{kT_e} \right\} \quad (I.3)$$

$$j_i = en_i v_i \quad (I.4)$$

A force balance on the ion yields

$$F = -eE = m \frac{dv_i}{dt} = mv_i \frac{dv_i}{dy} \quad (I.5)$$

Substituting for v_i yields

$$E = \frac{m_i j_i}{2e^3 n_i} \frac{d \left\{ \frac{j_i}{n_i} \right\}}{dy} \quad (I.6)$$

or

$$E = \frac{-m_i}{e^3} \frac{j_i}{n_i^2} \left\{ \frac{dj_i}{dy} - \frac{j_i}{n_i} \frac{dn_i}{dy} \right\} \quad (I.7)$$

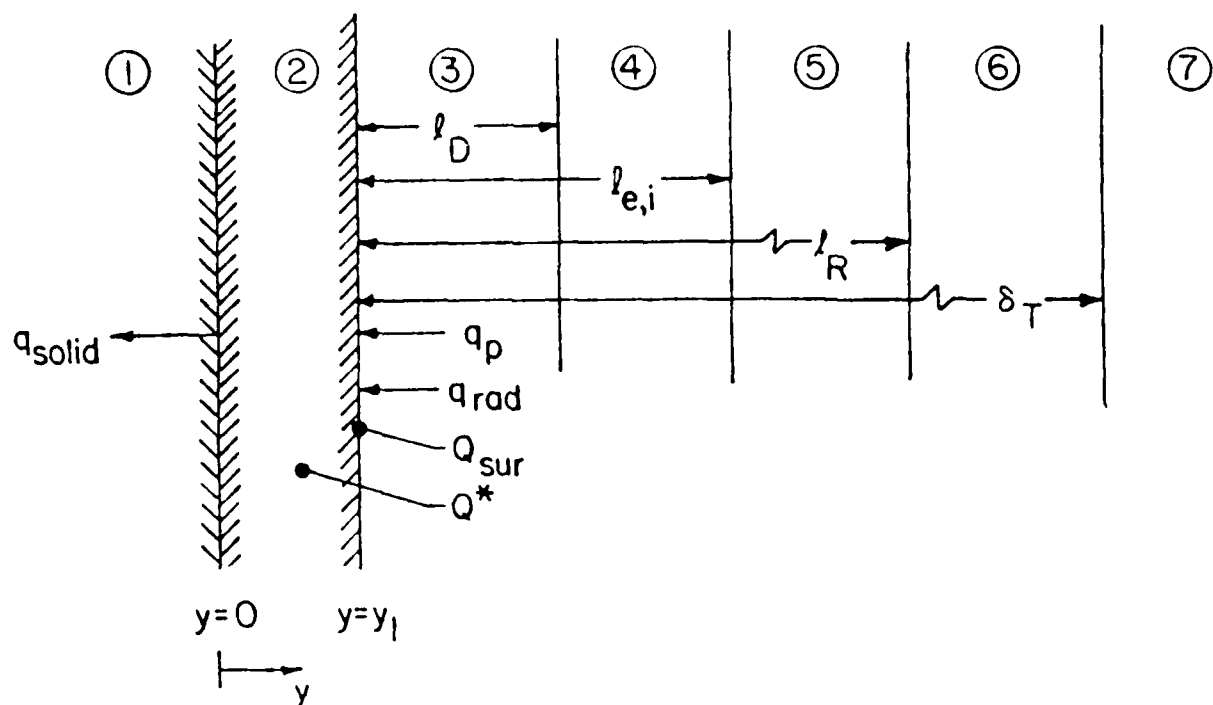


Figure 1.

Schematic of model for energy balance in the vicinity of a cathode of a MPD engine in the absence of an applied magnetic field. (1) solid conductor; (2) conductor in "mushy" state; (3) Debye region; (4) collisionless region outside sheath; (5) Region within a recombination length; (6) thermal boundary layer; (7) free stream.

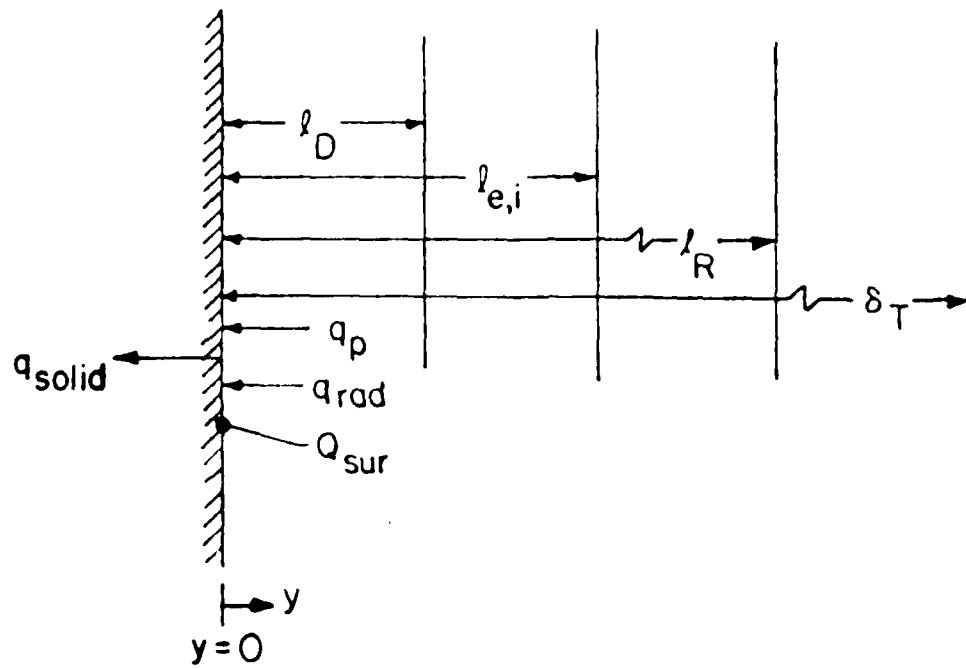


Figure 2. Schematic of a simplified model.

Planned Solution Procedure

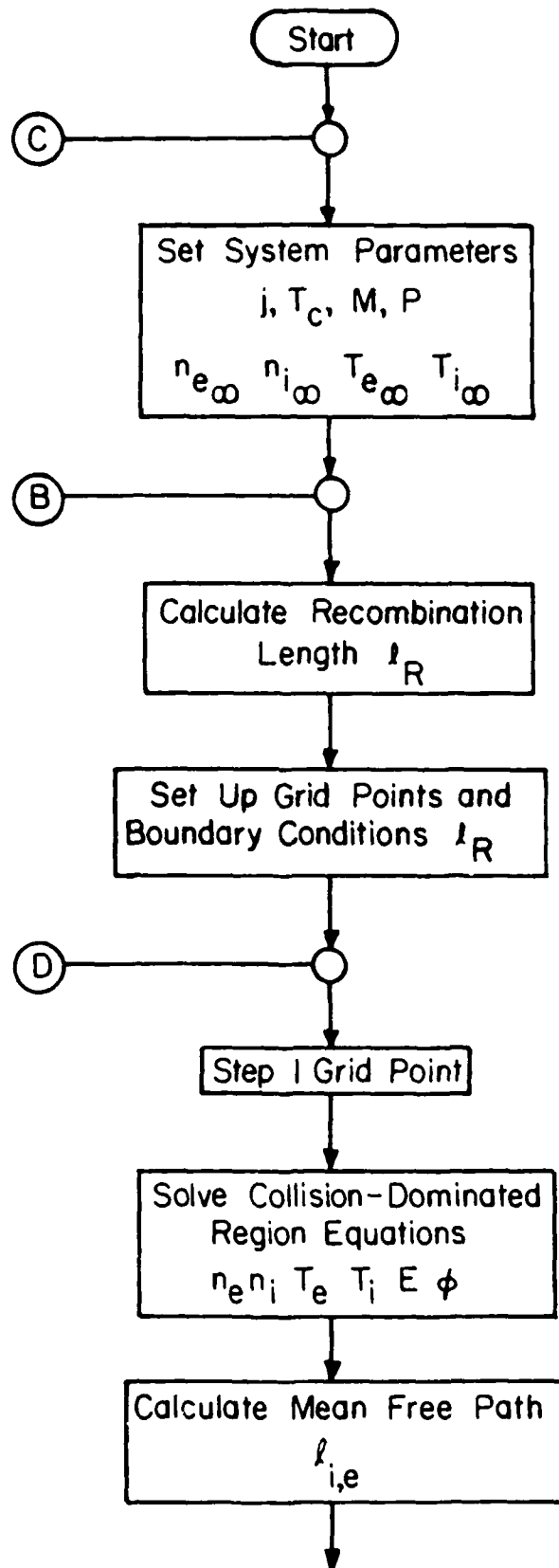


Figure 3. Solution procedure.

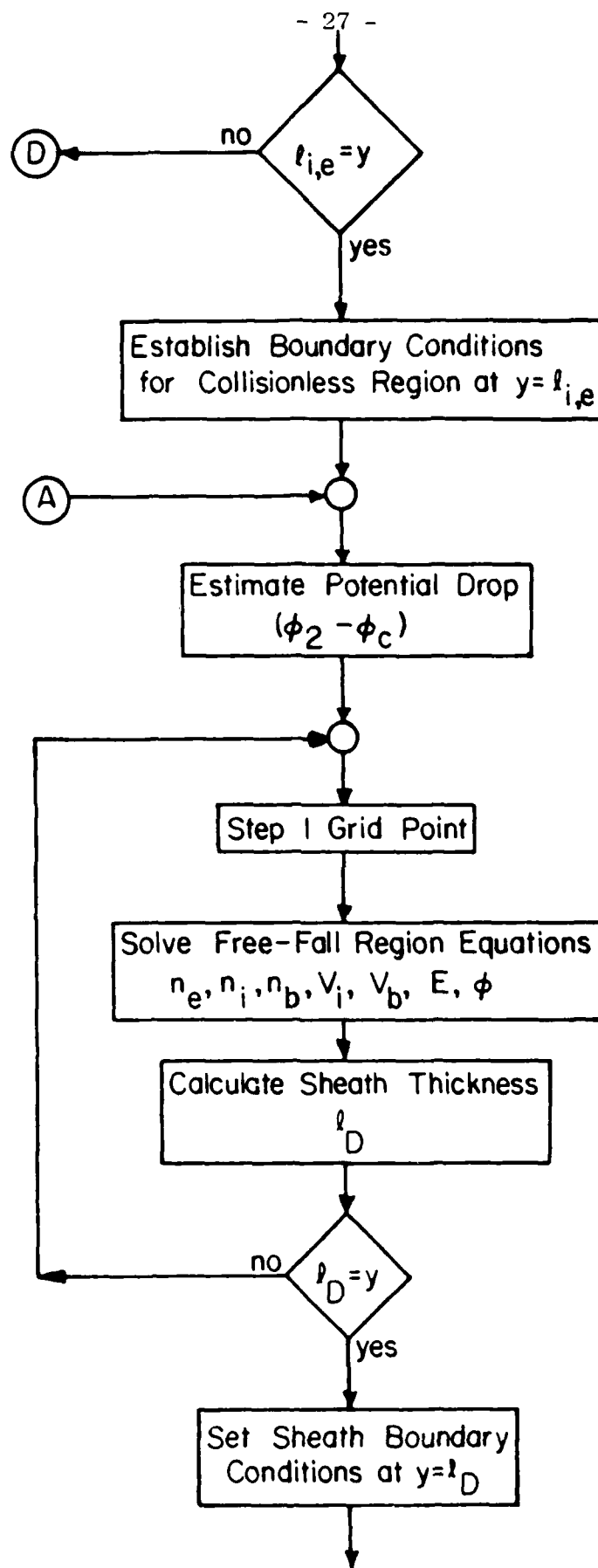


Figure 3. Continued

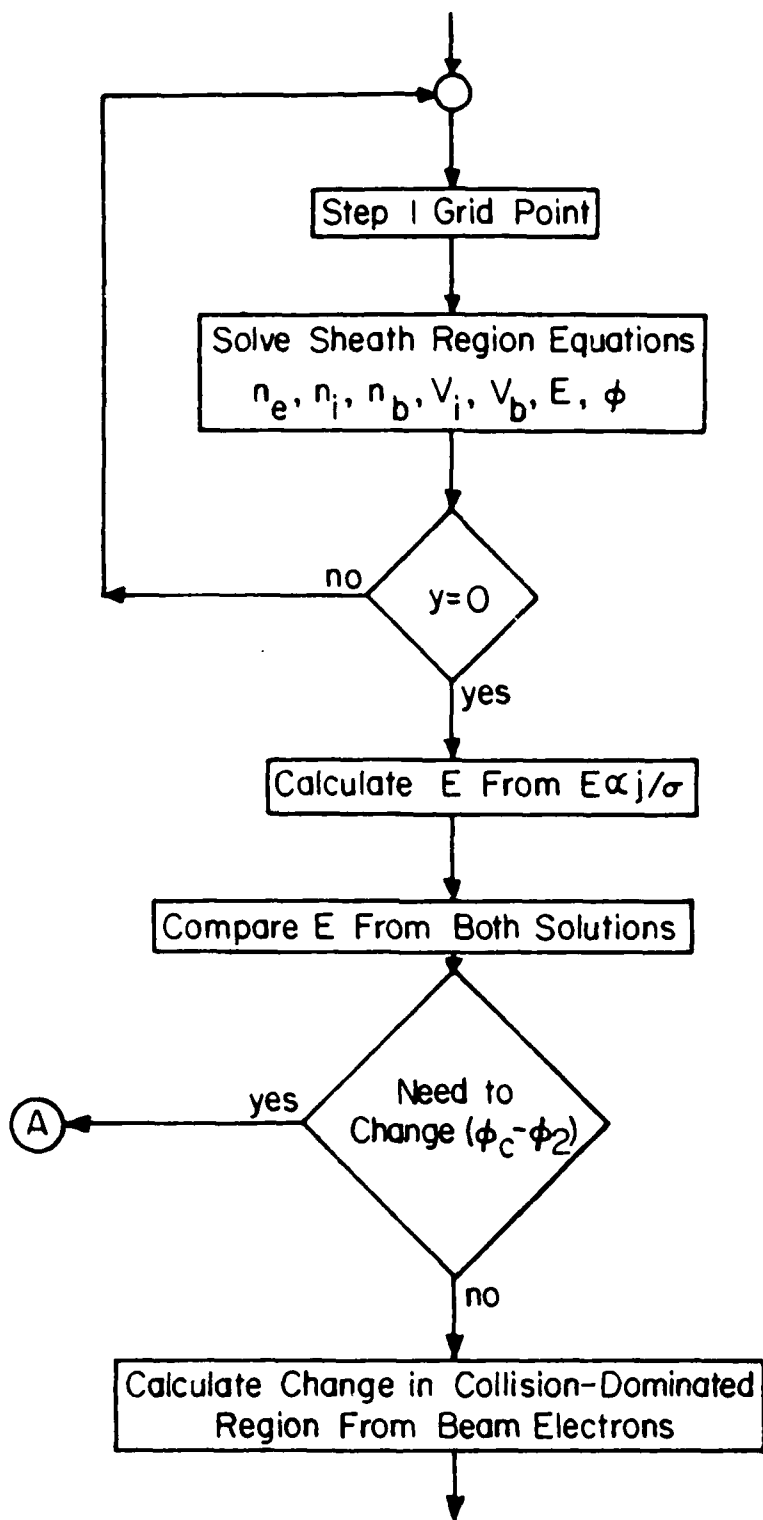


Figure 3. Continued

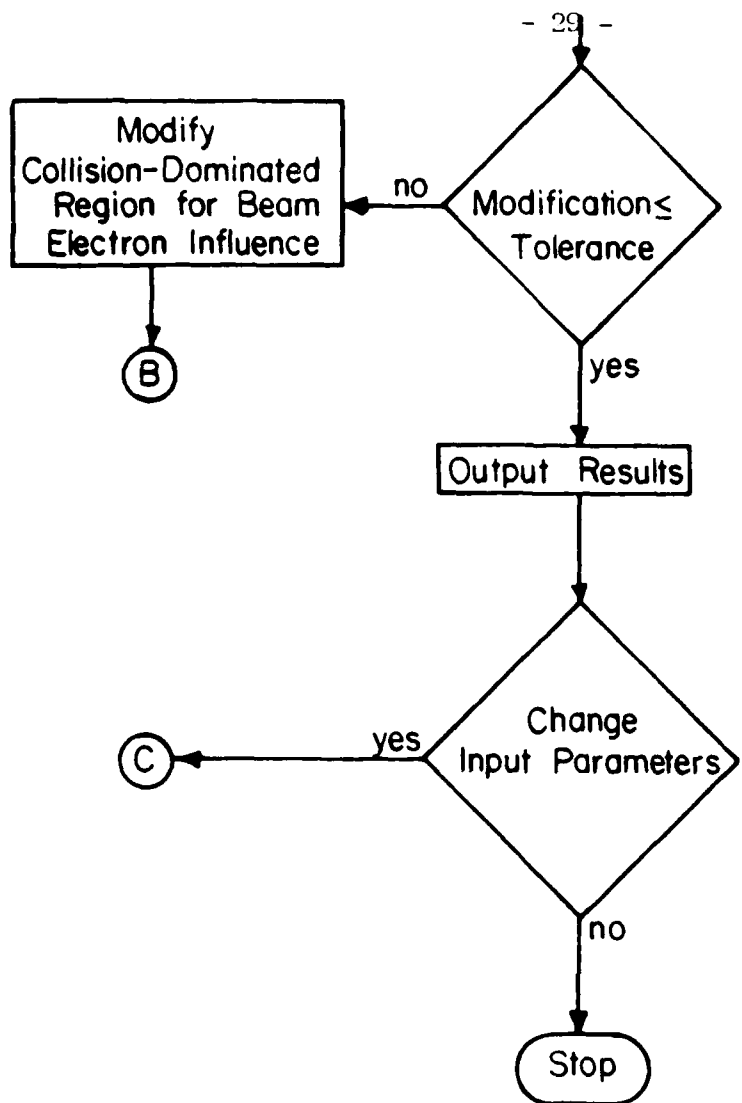


Figure 3. Concluded

STABILIZABILITY ANALYSIS OF MPD THRUSTERS

1. INTRODUCTION

One of the main concerns of the control theory for systems governed by partial differential equations, is whether or not the system is stable, if unstable, how the system could be stabilized by applying a proper set of control inputs to the system. These systems which are often called distributed parameter systems, are represented by a set of states which are functions of time and spatial coordinate(s). At each instance of time, every state of the system has a distribution on the spatial plane (i.e. the state belongs to an infinite-dimension functional space). However, for finite dimensional systems governed by a set of ordinary differential equations, a state is defined by a scalar at every instance of time. For finite dimensional systems Lyapunov stability theorem has become an important vehicle in derivation of stability analysis of dynamics of the system. This approach attempts to make statements about stability of motion of a dynamic system without any knowledge of the solutions to its governing equations.

Although the development of Lyapunov's stability theorem for ordinary differential equations has been widely investigated, its application to solutions of partial differential equations (distributed parameter systems) has been limited. Many stability results for distributed parameter systems have been derived by use of approximation methods. These methods, in general, use reduction of the partial differential equations by discretization [1]^{*} or by a truncation of the modal expansion [2,3] to a finite large order set of ordinary differential equations. Such approaches may not give sufficient

conditions for system stability. Another advantage of Lyapunov's method over approximate methods is that Lyapunov's stability theorem can be applied to both linear and nonlinear systems.

The attempt to apply Lyapunov's method to partial differential equation (PDE) was made by Massera and Zubov [4,5]. Massera derived sufficient conditions for stability of equilibrium solutions (steady states) of system of PDE. The generalization of Lyapunov's stability theorem based on the existence of a Lyapunov functional was established by Zubov. He derived necessary and sufficient condition for the stability of invariant set of dynamical systems in general metric spaces. The application of Lyapunov stability theorem based on the work of Zubov has been investigated by several authors. Hsu [6] applied this theory to a nuclear reactor system. Wang [7] considered the stability of those evolution equations whose solutions involve a semigroup property. There are also many other applications which utilize Lyapunov functions directly to study special problems [8,9]. A completely rigorous and abstract approach to the theory of Lyapunov stability for infinite dimensional system was studied by [10 and 11]. Stability study of an equilibrium solution of a magneto-plasma-dynamic (MPD) system for the special case, where the plasma equilibrium velocity is zero was addressed by the authors [12]. During the past budget year, the study has been focused on the problem of controlling the general equilibrium state of MPD thrusters. Moreover, the requirements for stability of a general equilibrium state of the system with an initial perturbation is derived. This analysis provides the foundation for development of required control inputs which guarantee stability of the

* Argument of \square denotes the reference number.

MPD system perturbed from a general equilibrium state. The problem of stabilizability which is tied to the controllability property of a distributed parameter system has been studied in general by [13] and [14] in Hilbert and Banach spaces, respectively. Applications of these studies on specific systems are investigated by [15] and [16] for wave equations with boundary and internal control problems. The generalized approach to the problem is investigated and the results are presented in the following sections.

Section 2 presents some minimum mathematical preliminary background related to the treatment of the problem. The focal point in this section is to give the notion of equivalent norm and a theoretical basis of why the bilinear functional (i.e. Lyapunov functional) can be regarded as an equivalent norm. In section 3 the semigroup properties of solutions of systems of PDE's is studied, while in section 4 the significance and the relation of these properties to stability conditions is presented. In section 5 the simple models of MPD thrusters, as it is studied in [17] and [18], is derived. The control of equilibrium states of the MPD system, represented by the models to avoid singularity at choking and stability conditions for the convergence of perturbation to the equilibrium state have been studied and presented in sections 6 and 7, respectively.

The report is concluded by presenting two possible approaches, based on the theory of characteristics and selection of some energy function, which could lead to the determination of general system stabilizability requirements. These approaches can be regarded as basis for future research in applying the stabilizability and controllability to the MPD system.

2. PRELIMINARIES

This section provides the mathematical basis and preliminaries for the following sections. The objective is to provide the readers with a quick review of the required definitions and theorems.

Definition: Let X be a vector space over real or complex field F . A norm on X , denoted by $\| \cdot \|$ is a real-valued function on X with the following properties:

- a) $\|x\| > 0$ if $x \neq 0$ and $\|x\| = 0$ for $x = 0$ for all $x \in X$
- b) $\|\alpha x\| = |\alpha| \|x\|$, for $\alpha \in F$ ($F = \mathcal{R}$ or \mathcal{C})
- c) $\|x + y\| \leq \|x\| + \|y\|$, for x and $y \in X$

Norms can be constructed in different ways.

If

$$x = \{x_i \mid i=1,2,\dots,n\}$$

Then for infinite dimensional space X , $n \rightarrow \infty$ the following is defined.

$$l_p \text{ norm} \triangleq \| \cdot \|_p = \left(\sum_{i=1}^n x_i^p \right)^{1/p} < \infty, \text{ and } p \geq 1$$

$$l_2 \text{ norm} \triangleq \| \cdot \|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} < \infty$$

$$l_\infty \triangleq \| \cdot \|_\infty = \sup_i |x_i|$$

$$L_2 \text{ norm} \triangleq \|f\|_{L_2} = \left(\int_1^{\infty} |f(t)|^2 dt \right)^{1/2}, f(t) \in \text{functional space } X$$

L^p , L_∞ norm are defined correspondingly.

Based on the definition used for the norm, different types of normed linear spaces can be specified. The two most commonly used normed spaces are Hilbert and Banach spaces. Hilbert space is a normed linear space with the norm being defined as $\|\cdot\|_{L_2}$ or $\|\cdot\|_2$. Respectively, Banach spaces are defined with $\|\cdot\|_{L_\infty}$ or $\|\cdot\|_\infty$ norms.

Lyapunov stability theory evaluates the stability with respect to a norm. This theory is concerned primarily with dissipative property of operators, which can be related to inner product property of the space of state variables. Therefore the natural space for stability analysis would be space with inner product norm (i.e. Hilbert space). However, for the study of stability in general Banach space, norm can be defined in term of semi-inner product. Since the stability properties are invariant under equivalent norms, the related definition and theorems are given in the following.

Definition: Let H_1 with norm $\|\cdot\|_1^2 = \langle \cdot, \cdot \rangle_1$ and H_2 with norm $\|\cdot\|_2^2 = \langle \cdot, \cdot \rangle_2$ be Hilbert spaces consisting of the elements of a linear vector space H and the given norms. The inner products are called equivalent if and only if their induced norms are equal. Hence by this notion of equivalent norm, as will be seen later, one can preserve the stability properties from one Hilbert space into another Hilbert space with equivalent norms.

Linear Functional: Every functional $T : x \rightarrow \epsilon$ is called a linear functional provided:

$$x_1, x_2 \in X \subset B \text{ (Banach Space)}$$

$$T(x_1) + T(x_2) = T(x_1 + x_2)$$

$$T(\alpha x) = \alpha T(x)$$

$$T(0) = 0$$

A linear functional T is bounded if

$$\|T\| = \sup_{x \in B} \frac{|T(x)|}{\|x\|} < \infty$$

Riesz Representation Theorem:

Every bounded linear functional in a Hilbert space H can be written in the form $\langle x, y_0 \rangle$ where $y_0 \in H$ is uniquely determined from $T(x)$.

$$T(x) = \langle x, y_0 \rangle \quad \forall x \in H$$

Lax-Milgram Theorem:

Let H be a Hilbert space and let $B(x, y)$ be a complex valued functional defined on the product Hilbert space $H \times H$ which satisfies the following conditions:

i. *Sesqui-linearity*, i.e.,

$$B\left((\alpha_1 x_1 + \alpha_2 x_2), y\right) = \alpha_1 B(x_1, y) + \alpha_2 B(x_2, y)$$

and

$$B\left(x, (\beta_1 y_1 + \beta_2 y_2)\right) = \bar{\beta}_1 B(x, y_1) + \bar{\beta}_2 B(x, y_2)$$

where $\bar{\beta}_1, \bar{\beta}_2$ are complex conjugates of β_1, β_2 respectively.

ii. *Boundedness*, i.e., there exists a positive constant γ such that

$$|B(x, y)| \leq \gamma \|x\| \cdot \|y\|$$

iii. *Positivity*, i.e., there exists a positive constant δ such that

$$B(x, x) \geq \delta \|x\|^2$$

Then there exists a uniquely determined bounded linear operator $S \in L(H, H)$ with a bounded linear inverse $S^{-1} \in L(H, H)$ such that

$$\langle x, y \rangle = B(x, Sy); \quad \|S\| < 1/\delta$$

and

$$\langle x, S^{-1}y \rangle = B(x, y); \quad \|S^{-1}\| < \gamma$$

This theorem can be used in relation with equivalent inner products and norms.

Theorem: Two inner products defined on a real linear vector space H are equivalent if and only if there exists a symmetric bounded positive definite linear operator $S \in L(H, H)$ such that

$$\langle x, y \rangle_2 = \langle x, Sy \rangle_1$$

where the indices identify the inner product in Hilbert spaces 1 and 2. From the fact that $B(x, y) = \langle x, Sy \rangle_1 = \langle x, y \rangle_2$, is sesqui-linear, bounded and positive due to the properties of S , it follows that $\langle x, y \rangle_2$ satisfies all the properties of an inner product. Hence the norms defined by these inner products are equivalent. Application of the equivalent norms, yields derivation of the stability properties of the general operators (A) in differential equations of the form

$$\dot{\underline{x}} = A\underline{x},$$

from the knowledge of the properties of the operator A . The concept of equivalent inner products can be extended to complex Hilbert spaces.

3. SEMI-GROUPS AND ABSTRACT EVALUATION EQUATIONS

Finite order systems can be described by a set of ordinary differential equations. The simplest systems are linear autonomus systems which can be formulated by

$$\dot{v} = Av \quad v(0) = v_0 \quad (3-1)$$

where $v: [0, T] \rightarrow \mathcal{R}^n$ is the solution of the system which is a vector of continuously differentiable functions. The matrix A is a linear operator which acts on $v \in \mathcal{R}^n$ (n-dimensional vector space), or $A \in \mathcal{L}(\mathcal{R}^n)$. For $v_0 \in \mathcal{R}^n$, $v = e^{At}v_0$ is the homogeneous solution which satisfies

$$\frac{d}{dt}(e^{At}v_0) = Ae^{At}v_0 = A(v)$$

where

$$e^A = I + A + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} + \dots$$

Usually external effects such as body forces can be modeled into the above mentioned system by a vector valued function $g: t \in [0, T] \rightarrow \mathcal{R}^n$

$$\dot{v} = Av + g(t) \quad v(0) = v_0 \quad (3-2)$$

Then the solution is

$$v(t) = e^{At}v_0 + \int_0^t e^{A(t-s)} g(s)ds \quad (3-3)$$

Usually the excitation term $g(t)$ can be represented in terms of a vector valued (input) function $u(t): [0, T] \rightarrow \mathcal{R}^m$ with $g(t) = Bu(t)$ where $B: \mathcal{R}^m \rightarrow \mathcal{R}^n$

Abstract Equations:

In the case of distributed parameter systems, the mathematical description is usually given by a set of partial differential equations. In order to generalize the results from the finite-dimensional systems, the set of partial differential equations can

be transformed into abstract form of equations (3-1) and (3-2). The state of system v belongs to some Banach space B or more commonly, to a Hilbert space H . In this case the abstract equations represent a set of infinite-dimensional states and the operator A can no longer be treated as a bounded operator. Because it is a differential operator. For finite-dimensional systems the following properties are considered.

- (a) $v(0; v_0) = v_0$ (i.e. state at time 0, starting with initial data v_0)
- (b) $v(t_1 + t_2; v_0) = v \left(t_1; v(t_2; v_0) \right) = v \left(t_2; v(t_1; v_0) \right)$
- (c) $v(t, v_0)$ is continuous in t and v_0 (i.e. if one of them changes slightly, then the solution should not be changed drastically).

Based on abstraction of the above conditions, the following definition can be given.

Definition:

A (strongly continuous) semigroup is an operator $T(t): R^+ \rightarrow \mathcal{L}(V)$, where $\mathcal{L}(V)$ is the space of all bounded operators on V into V . The following properties characterize a semigroup

- (a) $T(0) = I$
- (b) $T(t_1 + t_2) = T(t_1)T(t_2)$
- (c) $\lim_{t \rightarrow 0} T(t)v = v$, for all $v \in V$ (i.e. T is strongly continuous at $t = 0$).

Theorem:

Let $T(t)$ be a semigroup on R^+ to $\mathcal{L}(V)$, where V is a Banach Space, then

(a) $\|T(t)\|$ is bounded on every compact interval of $[0, \infty]$ such that $\|T(t)\| \leq Me^{\omega t}$ for some $M, \omega \in \mathbb{R}$.

(b) $T(t)$ is strongly continuous, on $[0, \infty]$.

Definition: The infinitesimal generator of a strongly continuous semigroup $T(t)$ is defined by A , when

$$Av = \lim_{t \rightarrow 0} \left\{ \frac{1}{t} (T(t)v - v) \right\}$$

$$v \in D(A) \subseteq V$$

It should be noted that in general, A , will be an unbounded operator. The operator A is closed (i.e. its range and domain coverage to some element of their respective spaces), and its domain covers the closure of space V (i.e. it is dense in V).

Theorem: Let $T(t)$ be a strongly continuous semigroup on a Banach space V with infinitesimal generator A . If $v_0 \in D(A)$, then

(a) $T(t)v_0 \in D(A)$ for $t \geq 0$

(b) $\frac{d}{dt}(T(t)v_0) = A(T(t)v_0) = T(t)Av_0, t > 0$

(c) $\frac{d^n}{dt^n}(T(t)v_0) = A^n(T(t)v_0) = T(t)A^n v_0$ for $v_0 \in D(A^n); t > 0$

(d) $T(t)v_0 - v_0 = - \int_0^t T(s)Av_0 ds; t > 0$

(e) $D(A^n)$ is dense in V for $n = 1, 2, \dots$ and A is closed.

For finite dimensional system, Laplace transform of e^{At} is $L(e^{At}) = (sI - A)^{-1}$. This can be generalized to semigroups by the following proposition

Proposition: If $T(t)$ is a strongly continuous semigroup with infinitesimal generator A , then $\operatorname{Re}(s) > \omega$ for $s \in \rho(A)$

where, $\rho(A) \triangleq$ resolvent set of A is defined by

$$\rho(A) = \left\{ s; (sI - A)^{-1} \in \mathcal{L}(V); \text{ bounded linear operator on } V \right\}.$$

Note that ω is given by

$$\omega = \inf \left\{ \omega; \|T(t)\| \leq Me^{\omega t}; M, \omega \in \mathbb{R} \right\}.$$

Hille-Yoshida Theorem:

If:

(a) A is a closed linear operator on V such that $D(A)$ is dense in V .

(b) $(\lambda I - A)^{-1}$ exists for $\lambda > \omega$ where $\lambda, \omega \in \mathbb{R}$.

(c) $\|(\lambda I - A)^{-m}\| \leq \frac{M}{\lambda - \omega^m}$; $\lambda > \omega$; $m = 1, 2, \dots$

then A generates a strongly continuous semigroup $T(t)$ with the norm $\|T(t)\| \leq Me^{\omega t}$.

Definition: Consider strongly continuous semigroup $T(t)$ with

$$\|T(t)\| \leq Me^{\omega t} \quad M > 0, \omega < \infty$$

if $\omega = 0$ then $\|T(t)\| \leq M$ and $T(t)$ is called an equi-bounded semi-group with $t \geq 0$.

Definition: For equi-bounded semigroup when $M = 1$,

$$\|T(t)\| \leq 1, t \geq 0,$$

then $T(t)$ is called contraction semigroup.

In the stability theory of semigroups the contraction semigroups are very important. These contraction semigroups are closely related to the dissipative

property of infinitesimal generator A of the semigroup T . To study dissipative property of operator A one needs to apply it to inner products, hence Hilbert space would be a natural space for the study of dissipative property of operators.

Definition: An operator A defined on $D(A) \subseteq H$ (Hilbert Space) is dissipative if $\operatorname{Re} \langle Av, v \rangle \leq 0$ for every $v \in D(A)$.

Phillips and Lumer Theorem

Let A be a linear operator with domain $D(A)$ and range of A $[\operatorname{Ran}(A)]$ both in the Hilbert Space H where $\overline{D(A)} = H$ (i.e. domain of A densifies in H), then A generates a contraction semigroup on H if and only if A is dissipative with respect to the inner product defined on H , also $\operatorname{Ran}(I-A) = H$.

The proof is avoided as in earlier theorems, however, it should be mentioned that if the hypothesis of the above theorem holds, then one can derive

$$\left\| \frac{1}{(\lambda I - A)} \right\| \leq \lambda^{-1},$$

for all $\lambda > 0$, hence the Hille-Yosida theorem could be used. Namely, by setting $M = 1$, $\omega = 0$ one can find

$$\|T(t)\| \leq Me^{\omega t} = 1$$

Which is the fact that the operator A generates a contraction semigroup.

This result can be generalized to Banach Spaces if a Semi-inner product is used for inner product.

Corollary: If A is a closed linear operator with dense domain in H , then A generates a contraction semigroup if and only if A and A^* are dissipative. Note: A^* is the adjoint operator of A given by:

$$\langle x, Ay \rangle = \langle A^*x, y \rangle$$

So far the theory has covered the properties of an operator A with regard to semigroups. In general evolution equation, operator A can represent a hyperbolic or parabolic problem. However the specific problem which will be considered here is a hyperbolic system of the class

$$\dot{v} = Lv \quad v \in (L_2(\Omega), E^n)$$

The states form an Euclidean vector space (i.e. v is a vector valued function) where each element of the vector forms a function which belongs to infinite dimensional Hilbert space defined on one dimensional spatial region (domain) $\Omega = x \in [0, \ell]$. The operator L is similar to the operator A discussed before, and is unbounded with specific form:

$$Lv = A \frac{\partial v}{\partial x} + Bv$$

It is possible to establish the solution of this system of evolution equations in terms of semigroups using a theorem to be seen later. The solutions of this general system, depending on the conditions specified by the theorem, can result in either groups or semigroups.

4. STABILITY ANALYSIS FOR DISTRIBUTED PARAMETER SYSTEMS

In general, stability theory is an attempt to make some assessments about the motion of a system without apriori knowledge of the solutions to its governing dynamic equation. This phenomena which is an intrinsic property of a system in a heuristic sense is related to boundedness of the motion of the system about some prescribed regions. In order to define stability property for a generalized dynamical system, it is appropriate to give definitions of such systems and their invariant sets.

Definition: Let C be a closed subset of a complete metric space. A dynamical system on C is a family of maps S such that the following conditions hold:

- (a) For any finite value of $v(0)$, the vector function $v(t, v(0))$ is defined for all $t \in (-\infty, +\infty)$ and $v(0, v(0)) = v(0)$.
- (b) The vector function $v(t, v(0))$ is continuous in the aggregate of its arguments.
- (c) For all values of t and τ :

$$v(t + \tau, v(0)) = v(t, v(\tau, v(0)))$$

Hence it can be seen that in general a dynamical system forms a nonlinear group S on C . For a dynamical system on C , the positive semi-orbit through v is the set

$$\gamma(v) = \left\{ S(t)v : t \geq 0 \right\}.$$

This set defines the trajectory of solution with respect to initial condition v . The metric on the space C can be assigned as the norm between two states $v|_{t_2}$ and $v|_{t_1}$ as:

$$d(v|_{t_2}, v|_{t_1}) = \| v|_{t_2} - v|_{t_1} \|$$

where $v|_t$ for a distributed parameter system is a set of real valued functions at time t .

$$v|_t = \left\{ v_i(t, x) \right\} \quad i=1, \dots, n \text{ and } x = \text{spatial coordinate}$$

or ,

$$v|_t = S(t) v_o$$

when $d(v|_{t_2}, v|_{t_1}) = 0$ the state functions are identical.

Definition: An equilibrium state v_{eq} of a dynamical system is an element of the state space C such that $d(S(t)v_{eq}, v_{eq}) = 0$ for all $t \geq 0$. The set of all equilibrium states will be called the equilibrium set \mathcal{E} .

Definition: An invariant set of dynamical system is a subset $K \subset C$ such that for any initial state $v_o \in K$, the trajectory $v = S(t)v_o$ for all $t \geq 0$ remains in the set K , i.e., if $v_o \in K$, $(v = S(t) v_o) \in K$. For example $\left\{ v_{eq} \right\}$ and \mathcal{E} are invariant sets of the system.

The distance (metric) of a state v from an invariant set K is defined by

$$d(v, K) = \inf_{v' \in K} d(v, v')$$

Definition: An invariant set K of a dynamical system is stable with metrics d , if for every $\epsilon > 0$ there exists a $\delta(\epsilon, t_o)$ such that if $d(v_o, K) < \delta(\epsilon, t_o)$ then $d(S(t) v_o, K) < \epsilon$ for all $t \geq t_o$.

The invariant set is asymptotically stable if $d(S(t)v_o, K) \rightarrow 0$ as $t \rightarrow +\infty$

Stability Theorem for Distributed Parameter System

Zubov Theorem I: A necessary and sufficient condition for an invariant set K of a distributed parameter dynamical system v defined on space C to be stable is that there exists a real functional $L(v)$ having the following properties:

- (a) $L(v)$ is defined for all $t \geq 0$ and all in some region with $d(v, K) < r$.
- (b) For each sufficiently small real $\eta_1 > 0$ there exists a real $\eta_2 > 0$ such that $L(v) > \eta_2$ for $d(v, K) > \eta_1$, i.e. $L(v)$ is positive definite with respect to the norm $\|v\|$.
- (c) $\frac{d}{dt} L(v) \leq 0$ for $t \geq 0$, i.e. $L(v(t))$ is non-increasing with respect to all $t \geq t_0$, for $d(v_0, K) \leq \delta_0$, where δ_0 is a sufficiently small positive number.

Zubov's Theorem (II): An invariant set of K of a dynamic system on space C is asymptotically stable if and only if there exists a real functional $L(v)$ having the properties (a), (b), (c) of theorem I and,

- (d) $\frac{d}{dt} L(v) \rightarrow 0$ as $t \rightarrow +\infty$, provided that $d(v_0, K) \leq \delta_0$, where δ_0 is a sufficiently small real positive number.

Remark: In the Zubov Theorem, the assumption that the solutions v satisfy the properties of a dynamical system is needed to prove the necessary part of the theorem. For the general system, the existence of Lyapunov functional provides the sufficiency condition.

Stability of Homogeneous Linear Systems

Consider the abstract linear evolution equation of the previous section;

$$\dot{v} = Av \quad v(0) = v_0 \quad (4-1)$$

where the operation A generates a semigroup $T(t)$ on Banach Space V . For this system, one can define a stronger criterion for the stability by defining the notion of exponential stability.

Definition: The system with the above evolution equation (4-1) is exponentially stable if there exists real positive M and ω such that

$$\|T(t)\| \leq M e^{-\omega t}, t \geq 0.$$

Note that exponential stability implies asymptotic stability but the converse is not always true.

Clearly a linear system is exponentially stable if and only if its spectrum $\sigma(A)$ is contained in the left half of complex plane such that

$$\sup \operatorname{Re} \sigma(A) < 0$$

where spectrum is defined.

$$\sigma(A) = \left\{ \lambda: (\lambda I - A)v = 0, \text{ which implies } (\lambda I - A) \text{ is not one one} \right\}.$$

Hence $-\omega < 0$ can be chosen as $\sup \operatorname{Re} \sigma(A)$ and $\|T(t)\| \leq M e^{-\omega t}$.

Because some operators do not have eigenvalues then spectral analysis of operators in determination of their spectrum is not always achievable. In these cases the Lyapunov method can be adopted to analyse the stability of the solutions.

Lyapunov Method for Stability of Linear Systems

Let operator A in the evolution equation (4-1) be generator of a semigroup $T(t)$, then the null solution of (4-1) is asymptotically stable if there exists a Lyapunov functional $L(v)$ such that $L(v) > 0$ and $\dot{L}(v) \leq -\gamma \|v\|^2$ for $v \in D(A)$.

If the $\|\cdot\|$ is a Hilbert norm with $\overline{D(A)} = H$, then any Lyapunov functional with bilinear form $L: (H \cdot H) \rightarrow \mathbb{R}$, by Lax-milgram theorem can be made into an inner product of the form: $L: (v \cdot w) \rightarrow \langle v, Sw \rangle$, where S is symmetric real bounded operator.

Hence an equivalent norm can be defined $\|v\|_2 = \langle v, v \rangle_2 = \langle v, Sv \rangle$ and $\alpha \langle v, v \rangle < \|v\|_2^2 < \beta \langle v, v \rangle$.

By this equivalence relation, the existence of Lyapunov functional would lead into existence of operator S with aforementioned properties. Then the resulting Lyapunov functional by hypothesis satisfies:

$$L(v) = L(v, v) = \langle v, Sv \rangle.$$

$$\begin{aligned} \dot{L}(v) &= \lim_{t \rightarrow 0} \left\{ L(T(t)v, T(t)v) - L(v, v) \right\} \cdot 1/t \\ &= \lim_{t \rightarrow 0} \left\{ L((T(t) + I)v, (T(t) - I)v) \right\} \cdot 1/t = 2L(v, Av) = 2\langle v, SAV \rangle = 2\langle v, Av \rangle_2 \end{aligned}$$

$$\dot{L}(v) = 2\langle v, Av \rangle_2 \leq -\gamma \|v\|^2 \leq -\frac{\gamma}{\alpha} \|v\|_2^2$$

Also:

$$\lambda \langle v, v \rangle_2 - \langle Av, v \rangle_2 = \langle (\lambda I - A)v, v \rangle_2 \leq \|(\lambda I - A)v\|_2 \|v\|_2$$

or

$$\langle v, v \rangle_2 \left[\lambda - \frac{\langle Av, v \rangle_2}{\langle v, v \rangle_2} \right] \leq \|(\lambda I - A)v\|_2 \|v\|_2^2$$

where from the hypothesis $\frac{\langle Av, v \rangle_2}{\langle v, v \rangle_2} \leq -\frac{\gamma}{2\alpha}$,

$$\|(\lambda I - A)\|_2^{-1} \leq \left[\lambda - \frac{\langle Av, v \rangle_2}{\langle v, v \rangle_2} \right]^{-1} \leq \left[\lambda + \frac{\gamma}{2\alpha} \right]^{-1}$$

$$\|(\lambda I - A)^{-1}\|_2 \leq \frac{1}{\lambda + \frac{\gamma}{2\alpha}}$$

From this relation if the use of Hille-Yoshida theorem is made, it can be shown that operator A is the generator of a semigroup $T(t)$ such that

$$\|T(t)\| \leq M e^{-\frac{\gamma}{2\alpha} t}$$

which is a condition for exponential asymptotic stability.

5. MODELING OF MAGNETO-PLASMA DYNAMIC THRUSTERS

Recent increase in space missions and construction of the space station has attracted attention to new alternatives to chemical propulsion systems. One such system is categorized and known as electrical propulsion thruster. In general, electrical rockets should be able to develop considerably higher specific impulses than chemical or nuclear ones. However, this gain in specific impulse requires massive energy conversion mechanism. To avoid this, the electrical rockets generally have low thrust for navigation in low gravitational fields.

The propellant of an electrical rocket consists of either charged particles which are accelerated by electrostatic forces, or a stream of electrically conducting fluid (plasma) which is accelerated by electromagnetic and/or pressure forces.

The electrical accelerators are divided into three main categories: electrostatic, electrothermal and electromagnetic accelerators.

Electromagnetic accelerators use the conductivity of the plasma propellant to create an electromagnetic accelerating force as a "body force" within the plasma. The main focus of this work is aimed at modeling and analysis of steady magnetogas-dynamic flow accelerators, among other categories of electromagnetic accelerators. Figure (5-1) shows a schematic of such a system. The flow of ionized gas enters the thruster and is subjected to an electric field E and a magnetic field B , which are perpendicular to each other and to the gas velocity. Electromagnetic acceleration process is an aggregate of effects from compressible gas dynamics, ionized gas physics, electromagnetic field theory and particle electrodynamics. The individual analytic complexity of each of these phenomenon adds to the level of difficulty in adequate theoretical model for this composite system. Analytical progress normally stems from

simplified models which preserve the essential physical aspects of the specific situation.

The description of the motion of plasma in terms of Maxwell-Boltzmann distribution function is too detailed to be useful for many practical problems in the electromagnetic acceleration process. In these cases the ionized gas medium can be considered as a continuum fluid whose macroscopic physical properties may be described by conservation laws and Maxwell equations. These governing equations for the motion of plasma will be gas dynamic equations with the interaction terms due to electromagnetic forces. With this approach a simplified model for the problem, depicted schematically in figure (5-1), can be derived. As shown in this figure the plasma is flowing through the constant area channel along the x-coordinate. The channel is formed by two opposite conducting walls connected to cathode and anode poles respectively. Between these walls an electric field $E(x)$ is maintained in a y-direction. Normal to this electric field is a magnetic field $B(x)$, applied in the z-direction. The model is considered to be one dimensional i.e. only variations in x-direction are considered. The bulk properties of the gas (plasma) is shared by all species contained in the plasma. Application of the conservation laws to the flow through an element of volume results in the following equations. From the continuity equation, one can find

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad (5-1)$$

where ρ and u are the density and the velocity of gas respectively. The momentum equation results in:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} = - \frac{\partial p}{\partial x} + F_e + F_v$$

where p is the sum of thermodynamic pressure and radiation pressure P_R , however, P_R

is assumed negligible. F_e is electromagnetic force (Lorentz force) per unit volume, i.e. $F_e = j \times B$ and F_v represents combination of viscous and collisional force and assumed to be negligible compared to other terms. After expansion of the momentum equation and substitution from continuity equation, the following is resulted.

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = - \frac{\partial p}{\partial x} + jB \quad (5-2)$$

Due to the one-dimensional assumptions, the variations in the y and z directions are negligible, and the velocity components in y and z directions would be deleted. From the energy equation for the element of volume one can derive the following:

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial(\rho u h)}{\partial x} = jE + \frac{\partial(u\tau)}{\partial x} + \frac{\partial Q}{\partial x}$$

where

e = specific total energy (internal energy + potential energy
+ kinetic energy)

h = specific total enthalpy ($h = e + P/\rho$)

τ = shear stress tensor

Q = heat flux due to convection and radiation.

On the right hand side of the above energy equation the term jE represents the Joule heating due to the application of electric field. This term can be considered as the dominant form of dissipation of energy. The plasma can be assumed as a perfect gas, with the state equation

$$P = \rho RT, \quad R = R_A/m$$

where temperature is denoted by T , and R_A and m are the gas constant of plasma and molecular weight of the plasma, respectively. As a result, the specific energy, e , and specific enthalpy, h , can be represented as

$$e = c_v T + u^2/2$$

$$h = c_p T + u^2/2$$

when the potential energy terms are negligible. The specific heat coefficients c_v and c_p can be considered

$$c_v = 3/2 R$$

$$c_p = 5/2 R$$

Hence the energy equation can be approximated and rewritten as follows:

$$\frac{\partial}{\partial t} \left[\rho c_v T + \rho \frac{u^2}{2} \right] + \frac{\partial}{\partial x} \left[\rho u c_p T + \rho \frac{u^3}{2} \right] = jE \quad (5-3)$$

The Maxwell equations can be written as

$$\nabla \times B_{\text{vec}} = \mu_e \vec{j} + \mu_e \frac{\partial \epsilon \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = - \frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

$$\nabla \cdot E = \frac{1}{\epsilon} \rho_e$$

For many practical problems one can show that the displacement current $\frac{\partial \epsilon \vec{E}}{\partial t}$ and excess electric charge ρ_e are negligible and that the energy in the electric field is much smaller than that in the magnetic field. As a result the set of Maxwell equations will become

$$\begin{aligned}\frac{1}{\mu_e} \cdot \nabla \cdot B &= j \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \cdot B &= 0 \\ \nabla \cdot E &= 0\end{aligned}\tag{5-4}$$

If one specifies that the magnetic field generated by the current flowing in the gas is negligible with respect to the applied field $B(x)$, then the electromagnetic field relations (set of equations 5-4) can be decoupled from the dynamics of the plasma.

By Ohm's law, one can relate current density j with the applied fields;

$$j = \sigma(E - uB), \quad \sigma = \sigma(\rho, T)$$

where σ is a transport coefficient and it is called the electrical conductivity.

After substitution for $\frac{\partial \rho}{\partial t}$, $\frac{\partial u}{\partial t}$ in the energy equation (5-3), one can arrive at the following set of equations (5-5 to 5-7) as the set of dynamical governing equations for the MPD thruster

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0\tag{5-5}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{R}{\rho} \frac{\partial(\rho T)}{\partial x} + \frac{jB}{\rho}\tag{5-6}$$

$$\frac{\partial T}{\partial t} + \frac{RT}{c_v} \frac{\partial u}{\partial x} + u \frac{\partial T}{\partial x} + \frac{j(E - uB)}{\rho c_v}\tag{5-7}$$

where j is given by Ohm's law.

6. CONTROL OF MPD THRUSTER AT EQUILIBRIUM STATE

In the set of non-steady equations derived in the previous section, i.e. equations (5-5 to 5-7), it can be seen that the parameters B and E can be regarded as input functions to the system. One of the crucial questions about the behavior of this system is how to characterize the relation between the system response and those inputs. The general response characterization of these systems which change with both time t and spatial coordinate x , is a complex problem. However, one can break the problem into steps, by first trying to make assessments about equilibrium states of the system of the partial differential equations. Hence, one can pose a question of what choice of inputs would steer the system to an equilibrium set of states, and under what conditions the steering would not be plausible. In order to answer these arguments about the equilibrium states of the system one has to derive the equilibrium set of equations from the original partial differential equations. If states of the system are represented by a vector valued function v , as follows

$$v = \begin{bmatrix} \rho(t,x) \\ u(t,x) \\ T(t,x) \end{bmatrix}$$

then the equilibrium vector would be denoted by v_e

$$v_e = \begin{bmatrix} \rho_e(x) \\ u_e(x) \\ T_e(x) \end{bmatrix}$$

Similarly the inputs (control) functions can be defined by a vector valued function

U ,

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} jB \\ jE \end{bmatrix}$$

At equilibrium, system states reach their steady state values and their time derivatives would be zero, i.e. $\frac{\partial v_e}{\partial t} = 0$, and equations (5-5 to 5-7) will turn to the following equations.

$$\frac{d}{dx} (\rho_e u_e) = 0 \quad \text{or} \quad \frac{d\rho_e}{dx} = -\frac{\rho_e}{u_e} \frac{du_e}{dx} \quad (6-1)$$

$$u_e \frac{du_e}{dx} + \frac{R}{\rho_e} \frac{d(\rho_e T_e)}{dx} = \frac{U_1}{\rho_e} \quad (6-2)$$

$$\frac{RT_e}{c_v} \frac{du_e}{dx} + u_e \frac{dT_e}{dx} = \frac{U_2 - u_e U_1}{\rho_e c_v} \quad (6-3)$$

Substituting equation (6-1) into (6-2) would result in the following equations,

$$\left[u_e - \frac{RT_e}{u_e} \right] \frac{du_e}{dx} + R \frac{dT_e}{dx} = \frac{U_1}{\rho_e}$$

$$\left[\frac{RT_e}{c_v} \right] \frac{du_e}{dx} + u_e \frac{dT_e}{dx} = \frac{U_2 - u_e U_1}{\rho_e c_v}$$

or,

$$\frac{du_e}{dx} = \frac{\gamma u_e U_1 - (\gamma-1)U_2}{\rho_e (u_e^2 - \gamma R T_e)} \quad (6-4)$$

$$\frac{dT_e}{dx} = \frac{-u_e^3 U_1 + (u_e^2 - R T_e) U_2}{\rho_e u_e c_v (u_e^2 - \gamma R T_e)} \quad (6-5)$$

where γ is the specific heat ratio c_p/c_v . The speed of sound a_e is defined by $a_e^2 = \gamma R T_e$, (6-4) and (6-5) can be rewritten into a vector differential equation form.

$$\frac{dv_e}{dx} = \frac{1}{u_e^2 - a_e^2} \begin{bmatrix} \frac{\gamma u_e}{\rho_e} & -\frac{\gamma-1}{\rho_e} \\ \frac{u_e^2}{\rho_e c_v} & \frac{u_e^2 - R T_e}{\rho_e u_e c_v} \end{bmatrix} U \quad (6-6)$$

$$U = \rho_e^2 c_v \begin{bmatrix} \frac{u_e^2 - RT_e}{\rho_e u_e c_v} & \frac{\gamma-1}{\rho_e} \\ \frac{u_e^2}{\rho_e c_v} & \frac{\gamma u_e}{\rho_e} \end{bmatrix} \frac{dv_e}{dx} \quad (6-7)$$

Therefore, for any equilibrium state configuration, it is possible to arrive at a control vector in the local sense with respect to the coordinate x . However, in transition from subsonic flow to supersonic flow, where $u_e = a_e$ (i.e. Mach number is unity), the control inputs U_1 and U_2 become dependent on each other, or one should choose U_1 and U_2 according to a certain relation such that the passage from subsonic to supersonic velocities and vice versa would be plausible.

$$U_2 = \frac{\gamma u_e U_1}{(\gamma-1)}, \text{ at } u_e = \sqrt{\gamma RT_e}$$

or

$$\frac{E}{B} = \frac{\gamma u_e}{(\gamma-1)}$$

This effect is the same as choking condition in gas dynamics which is extended to magneto gas dynamics [19].

Based on the definition of Mach number M as $M = \frac{u_e}{\sqrt{\gamma RT_e}}$, it can be shown that

$$M^2 = \frac{u_e^2}{\gamma RT_e} \text{ and } \frac{dM^2}{M^2} = \frac{du_e^2}{u_e^2} - \frac{dT_e}{T_e}$$

or

$$\begin{aligned} \frac{dM^2}{M^2} &= 2 \frac{du_e}{u_e} - \frac{dT_e}{T_e} \\ \frac{1}{M^2} \frac{d(M^2)}{dx} &= \frac{2}{u_e} \frac{du_e}{dx} - \frac{1}{T_e} \frac{dT_e}{dx} \end{aligned}$$

If the substitution for $\frac{du_e}{dx}$ and $\frac{dT_e}{dx}$ from equation (6-4) and (6-5) into the above equation is made, one could derive the following results

$$\frac{(1-M^2)}{M^2} \frac{dM^2}{dx} = \frac{\gamma M^2 + 1}{\rho_e u_e c_p T_e} U_2 - \frac{2 \left[\frac{(\gamma-1)}{2} M^2/2 + 1 \right]}{p_e} U_1 \quad (6-8)$$

For subsonic flow $M < 1$, increase in M is possible provided the following inequality is satisfied.

$$\frac{U_2}{U_1} > \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]} \frac{\rho_e u_e c_p T_e}{p_e} = \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]} \frac{\gamma u_e}{(\gamma-1)} \quad (6-9)$$

For supersonic flow $M > 1$, increase in M along the thruster occurs if,

$$\frac{U_2}{U_1} < \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]} \frac{\gamma u_e}{\gamma-1} \quad (6-10)$$

At sonic condition the ratio of control inputs $\frac{U_2}{U_1}$ should comply with the aforementioned quantity, $\left[\frac{\gamma u_e}{\gamma-1} \right]$. Clearly, for the decelerating flow (when M decreases along the flow) the direction of inequalities (6-9) and (6-10) would be reversed.

One approach to construct acceptable control inputs U_1 and U_2 , which is based on exclusion of singularity at choking, is to consider

$$U_2 = U_3 + \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]} \frac{\gamma u_e}{(\gamma-1)} U_1, \quad (6-11)$$

where U_3 is required to have the following properties for accelerating flow;

$$U_3(M) = \begin{cases} U_3 > 0 & \text{for } M < 1 \\ U_3 = 0 & \text{for } M = 1 \\ U_3 < 0 & \text{for } M > 1 \end{cases} \quad (6-12)$$

Substitution for U_2 from equation (6-11) and (6-12) in equations (6-4) and (6-5) would result in the following

$$\frac{du_e}{dx} = \frac{u_e U_1}{p_e(\gamma M^2 + 1)} - \frac{(\gamma - 1) U_3}{\gamma p_e(M^2 - 1)}$$

$$\frac{dT_e}{dx} = \frac{-2U_1}{\rho_e R(\gamma M^2 + 1)} + \frac{(M^2 - 1/\gamma) U_3}{\rho_e c_v u_e(M^2 - 1)}$$

In order to avoid singular (unacceptable) control distributions, one should consider U_3 a function with the following form;

$$U_3 = (1 - M^2)U_4 \quad (6-13)$$

with $U_4 > 0$ for accelerating flow, the requirements of (6-12) will be satisfied.

Substitution of (6-13) into (6-4) and (6-5) will result in the following equations.

$$\frac{du_e}{dx} = \frac{u_e U_1}{p_e(\gamma M^2 + 1)} + \frac{(\gamma - 1) U_4}{\gamma p_e} \quad (6-14)$$

$$\frac{dT_e}{dx} = \frac{-2U_1}{\rho_e R(\gamma M^2 + 1)} - \frac{(M^2 - 1/\gamma) U_4}{\rho_e c_v u_e} \quad (6-15)$$

Since the Joule heating (U_2) is a positive function, for supersonic flow, U_3 becomes negative and from condition (6-1) one would have;

$$0 < \frac{U_1}{\left[\frac{\gamma u_e U_1}{\gamma - 1} \right]} < \frac{[(\gamma - 1)M^2 + 2]}{(M^2 - 1)(\gamma M^2 + 1)} \quad \text{for } M^2 > 1. \quad (6-16)$$

For $M^2 \leq 1$, U_2 in equation 6-11 is positive for any $U_4 > 0$. Also, the current density j should be positive (in the chosen direction to accelerate the flow), hence from Ohm's

law one can find the following conditions,

$$J = \sigma(E - uB) > 0$$

$$E > uB \quad (6-17)$$

or

$$U_2 > u_e U_1$$

From equations (6-11) and (6-13), U_2 can be substituted into inequality (6-17), hence;

$$\text{for } M < 1 \quad \frac{U_4(1-M^2)}{\left[\frac{\gamma u_e U_1}{\gamma-1} \right]} + \frac{[(\gamma-1)M^2 + 2]}{(\gamma M^2 - 1)} - \frac{(\gamma-1)}{\gamma} > 0 \quad (6-18)$$

$$\text{for } M > 1 \quad \frac{U_4}{\left[\frac{\gamma u_e U_1}{\gamma-1} \right]} < \frac{[(\gamma-1)M^2 + 2]}{(\gamma M^2 + 1)(M^2 - 1)} - \frac{(\gamma-1)}{\gamma(M^2 - 1)} \quad (6-19)$$

It is clear that in subsonic flow, inequality (6-18) is always satisfied, hence there is no constraint on U_4 in this regime. However, for supersonic flow, inequality (6-19) represents a more restrictive constraint on U_4 than inequality (6-16). Hence a proper choice of U_4 can be selected to satisfy inequality (6-19). The steady state response can be found from equations (6-14) and (6-15) for some arbitrary choice of $U_1 > 0$ and by selection of U_4 according to the aforementioned process. It would be an interesting proposition to apply the theory of optimal control to find the optimal control inputs U_1 and U_4 among an arbitrary class of functions. Those optimal U_1, U_4 result in some minimized performance index or cost function. As an example, the cost function can be selected from the group of "fuel optimal" problems, i.e.,

$$IP = \int_{x=0}^{x=t} \left[U_4^2 + (u_{e \text{ out}} U_1)^2 \right] dx$$

Having in mind that the optimal control problem is involved with control constraint of the form of inequality (6-19) while U_4, U_1 are positive quantities.

Perturbation of nonlinear non-steady equations

Based on the equilibrium state (represented by $v_e(x)$), it would be interesting to evaluate the system behavior in the neighborhood of its equilibrium states. If the perturbation of states with respect to equilibrium is denoted by \hat{v} , then,

$$v = \hat{v} + v_e = \begin{bmatrix} \hat{\rho}(t, x) \\ \hat{u}(t, x) \\ \hat{T}(t, x) \end{bmatrix} + \begin{bmatrix} \rho_e(x) \\ u_e(x) \\ T_e(x) \end{bmatrix}$$

$$\hat{\rho} \ll \rho_e$$

$$\hat{u} \ll u_e$$

$$\hat{T} \ll T_e$$

By substitution of v into the dynamic equations (5-5) to (5-7), one would find;

$$\frac{\partial \hat{\rho}}{\partial t} + (\hat{\rho}u_e + \hat{u}\rho_e + \rho_e u_e + \hat{\rho}\hat{u}) = 0$$

$$\frac{\partial \hat{u}}{\partial t} + (u_e + \hat{u}) \frac{\partial(u_e + \hat{u})}{\partial x} + \frac{R}{\rho_e + \hat{\rho}} \frac{\partial(\rho_e T_e + \hat{\rho}T_e + \rho_e \hat{T} + \hat{\rho}\hat{T})}{\partial x} = \frac{U_1}{\rho_e + \hat{\rho}}$$

$$\frac{\partial \hat{T}}{\partial t} + \frac{R(T_e + \hat{T})}{c_v} \frac{\partial(u_e + \hat{u})}{\partial x} + (u_e + \hat{u}) \frac{\partial(T_e + \hat{T})}{\partial x} = \frac{U_2 - (u_e + \hat{u})U_1}{(\rho_e + \hat{\rho})c_v}$$

In order to proceed with easier notation, the perturbation states are denoted without " $\hat{}$ " sign and they should not be mistaken with their state functions. Moreover, the nonlinear terms are of negligible order in the sufficiently small neighborhood of equilibrium states. As an example, one observes

$$\hat{\rho} \frac{\partial \hat{u}}{\partial x} \ll \rho_e \frac{\partial \hat{u}}{\partial x}, \quad \hat{\rho}u_e \frac{\partial \hat{u}}{\partial x} \ll \rho_e u_e \frac{\partial \hat{u}}{\partial x}$$

similar statements are true about other nonlinear terms, hence combining with steady state equations (6-1) and (6-3) one would get.

$$\frac{\partial \rho}{\partial t} + u_e \frac{\partial \rho}{\partial x} + \rho \left[\frac{du_e}{dx} \right] + \rho_e \frac{\partial u}{\partial x} - \left[\frac{\rho_e}{u_e} \frac{du_e}{dx} \right] u = 0 \quad (6-20)$$

$$\begin{aligned} \frac{\partial u}{\partial t} + \left[\frac{RT_e}{\rho_e} \right] \frac{\partial \rho}{\partial x} + \left[\frac{u_e}{\rho_e} \frac{du_e}{dx} \right] \rho + \left[\frac{R}{\rho_e} \frac{dT_e}{dx} \right] \rho \\ + u_e \frac{\partial u}{\partial x} + \left[\frac{du_e}{dx} \right] u + R \frac{\partial T}{\partial x} - \left[\frac{R}{u_e} \frac{du_e}{dx} \right] T = 0 \end{aligned} \quad (6-21)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + \left[(\gamma-1) \frac{T_e}{\rho_e} \frac{du_e}{dx} \right] \rho + \left[\frac{u_e}{\rho_e} \frac{dT_e}{dx} \right] \rho + \left[\frac{U_1}{\rho_e c_v} \right] u \\ + \left[\frac{RT_e}{c_v} \right] \frac{\partial u}{\partial x} + \left[\frac{du_e}{dx} \right] T + u_e \frac{\partial T}{\partial x} = 0 \end{aligned} \quad (6-22)$$

This set of perturbed equations are linear and their characteristics depend on both equilibrium states and their derivatives.

7. LYAPUNOV STABILITY FOR HYPERBOLIC DISTRIBUTED PARAMETER SYSTEMS

In this section the Lyapunov stability theorem is modified and applied to the system of partial differential equations of (6-20) to (6-22). These equations represent the dynamics of perturbation state v about an equilibrium state vector v_e . Hence by Lyapunov method one can find whether any deviation from the equilibrium state is stable or it grows unboundedly with time. To be consistent with the notation of sections 3 and 4 one could rearrange equation (6-20) to (6-22) to the canonical form of the evolution equation. Here the notation $(\cdot)'$ is used to represent $\frac{d(\cdot)}{dx}$:

$$\frac{\partial v}{\partial t} = - \begin{bmatrix} u_e & \rho_e & 0 \\ \frac{RT_e}{\rho_e} & u_e & R \\ 0 & \frac{RT_e}{c_v} & u_e \end{bmatrix} \frac{\partial v}{\partial x} + \begin{bmatrix} u'_e & -\frac{\rho_e}{u_e} u'_e & 0 \\ \frac{u_e}{\rho_e} u'_e + \frac{R}{\rho_e} T'_e & u'_e & -\frac{R}{u_e} u'_e \\ (\gamma-1) \frac{T_e}{\rho_e} u'_e + \frac{u_e}{\rho_e} T'_e & \frac{U_1}{\rho_e c_v} + T'_e & u'_e \end{bmatrix} v \quad (7-1)$$

It is clear that $v = 0$ (i.e., null solution) represents the condition where there is no variation from the equilibrium state.

The characteristic directions (eigenvalues) of this system of equations are:

$$\lambda_1 = u_e, \quad \lambda_2 = u_e + \sqrt{\gamma RT_e}, \quad \lambda_3 = u_e - \sqrt{\gamma RT_e},$$

Therefore, (7-1) represents a system of hyperbolic partial differential equations. For a general hyperbolic system of the form

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + B v = L v \quad (7-2)$$

with v being a vector valued function of dimension n . The following theorem indicates that the operator L is the generator of a semigroup on $\overline{D(L)} = L^2(\{x \in [0, \ell] \mid v \in E^n\})$.

$$L v = A \frac{v}{x} + B v \quad (7-3)$$

Theorem 7-1: The linear operator L defined by (7-3) with A being symmetric is the generator of semigroups (groups if the number of positive eigenvalues is equal to the number of the negative eigenvalues, i.e., $n/2$) for the solution of systems of equations (7-2).

It is possible to show that the system of equation (7-1) can be made into the form of (7-2) with a symmetric A matrix.

For simplicity $\frac{\rho(\cdot)}{\rho_e}$ and $\frac{u(\cdot)}{u_e}$ are denoted by subscripts $(\cdot)_t$ and $(\cdot)_x$ respectively. As shown below, by dividing equation (6-20) by (ρ_e) , (6-21) by $(\sqrt{RT_e})$ and (6-22) by $(\sqrt{\gamma-1} T_e)$ the following symmetric A matrix can be obtained.

$$0 = \begin{bmatrix} \frac{\rho_t}{\rho_e} \\ \frac{u_t}{\sqrt{RT_e}} \\ \frac{T_t}{T_e \sqrt{\gamma-1}} \end{bmatrix} + \begin{bmatrix} u_e & \sqrt{RT_e} & 0 \\ \sqrt{RT_e} & u_e & \sqrt{(\gamma-1)RT_e} \\ 0 & \sqrt{(\gamma-1)RT_e} & u_e \end{bmatrix} \begin{bmatrix} \frac{\rho_x}{\rho_e} \\ \frac{u_x}{\sqrt{RT_e}} \\ \frac{T_x}{\sqrt{(\gamma-1) T_e}} \end{bmatrix} + \begin{bmatrix} u'_e & -\sqrt{RT'_e} \frac{u'_e}{u_e} & 0 \\ \frac{(u_e u'_e + RT'_e)}{\sqrt{RT_e}} & u'_e & -\sqrt{(\gamma-1)RT_e} \frac{u'_e}{u_e} \\ \frac{(\gamma-1)T_e u'_e + u_e T'_e}{\sqrt{\gamma-1} T_e} & \frac{\sqrt{R} \left[\frac{U_1}{\rho c_v} + T'_e \right]}{\sqrt{(\gamma-1)T_e}} & u'_e \end{bmatrix} \begin{bmatrix} \frac{\rho/\rho_e}{\sqrt{RT_e}} \\ \frac{u}{\sqrt{RT_e}} \\ \frac{T}{\sqrt{\gamma-1} T_e} \end{bmatrix} \quad (7-4)$$

It can be shown that

$$\begin{bmatrix} \frac{\rho_x}{\rho_e} \\ \frac{u_x}{\sqrt{RT_e}} \\ \frac{T_x}{T_e \sqrt{\gamma-1}} \end{bmatrix} = \begin{bmatrix} \left(\frac{\rho}{\rho_e} \right)_x + \frac{\rho \rho'_e}{\rho_e^2} \\ \left(\frac{u}{\sqrt{RT_e}} \right)_x + \frac{1}{2} \frac{u T'_e}{T_e^{3/2} \sqrt{R}} \\ \left(\frac{T}{T_e \sqrt{\gamma-1}} \right)_x + \frac{T}{\sqrt{\gamma-1}} \frac{T'_e}{T_e^2} \end{bmatrix} \quad (7-5)$$

The vector function v is then redefined by following new vector:

$$v = \begin{bmatrix} \frac{\rho}{\rho_e} \\ \frac{u}{\sqrt{RT_e}} \\ \frac{T}{T_e \sqrt{\gamma-1}} \end{bmatrix} \quad (7-6)$$

From substitution of (7-5) into (7-4) it results in;

$$v_t + A v_x + B v = 0 \quad (7-7)$$

where

$$A = \begin{bmatrix} u_e & \sqrt{RT_e} & 0 \\ \sqrt{RT_e} & u_e & \sqrt{(\gamma-1)RT_e} \\ 0 & \sqrt{(\gamma-1)RT_e} & u_e \end{bmatrix} = A^T \quad (7-8)$$

and

$$B = \begin{bmatrix} u'_e + \frac{\rho'_e}{\rho_e} u_e & \sqrt{RT_e} \left(-\frac{u'_e}{u_e} + \frac{1}{2} \frac{T'_e}{T_e} \right) & 0 \\ \frac{(u_e u'_e + RT'_e)}{\sqrt{RT_e}} + \sqrt{RT_e} \frac{\rho'_e}{\rho_e} & u'_e + \frac{1}{2} u_e \frac{T'_e}{T_e} & \sqrt{(\gamma-1)RT_e} \left(\frac{-u'_e}{u_e} + \frac{T'_e}{T_e} \right) \\ \frac{(\gamma-1)T_e u'_e + u_e T'_e}{T_e \sqrt{\gamma-1}} & \sqrt{(\gamma-1)RT_e} \left[\frac{1}{2} \frac{T'_e}{T_e} + \frac{\left(\frac{U_1}{\rho_e c_v} + T'_e \right)}{(\gamma-1)T_e} \right] & u'_e + \frac{u_e T'_e}{T_e} \end{bmatrix} \quad (7-9)$$

Construction of Lyapunov functional

To construct a Lyapunov functional it is sufficient to select the functional as an equivalent norm of a Hilbert space of states of equations (7-7), i.e., from the discussion in sections 2 and 4 this functional would be a bilinear form as follows

$$\mathcal{L} = (v, v)_2 = (v, Sv) = \int_0^l v^T S(x) v \, dx \quad (7-10)$$

where S is symmetric positive definite and bounded linear operator. In order to satisfy the sufficiency condition of Zubov's theorem for the stability of system of (7-2) one has to show that there exists a $S(x)$ for which $\frac{d\mathcal{L}}{dt} \leq 0$.

Equation (7-10) yields:

$$\frac{d\mathcal{L}}{dt} = \int_0^l \dot{v}^T S(x) v \, dx + \int_0^l v^T S(x) \dot{v} \, dx$$

Conjugate operators are defined based on bilinear form operation as following

$$z, Sy = (S'z, y), \quad S' = \text{conjugate of } S.$$

Here, since S is a real operator $S' = S^T$

$$\int_0^l v^T S(x) \dot{v} dx = \langle v, S\dot{v} \rangle = \langle S^T v, \dot{v} \rangle = \int_0^l (S^T v)^T \dot{v} dx$$

Therefore,

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= \int_0^l \dot{v}^T S(x) v dx + \int_0^l (S^T(x) v)^T \dot{v} dx \\ &= \langle v, S\dot{v} \rangle + \langle S^T v, \dot{v} \rangle \end{aligned}$$

The operator $S(x)$ is chosen to be symmetric $S = S^T$, then

$$\frac{d\mathcal{L}}{dt} = 2 \langle Sv, \dot{v} \rangle = 2 \int_0^l v^T S(x) \dot{v} dx \quad (7-11)$$

Now, from equations (7-7), one can substitute for \dot{v} (i.e., v_t) into (7-11). This results in the following.

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= 2 \langle Sv, (-Av_x) \rangle + 2 \langle Sv, (-Bv) \rangle \\ \frac{d\mathcal{L}}{dt} &= -2 \int_0^l v^T S(x) [A v_x] dx - 2 \int_0^l v^T S(x) [B v] dx \end{aligned}$$

where,

$$\begin{aligned} \int_0^l v^T S(x) A \frac{\partial v}{\partial x} &= \int_0^l \frac{\partial}{\partial x} [v^T S(x) A v] dx - \int_0^l v^T \left[\frac{\partial S(x)}{\partial x} \right] A v dx - \int_0^l v^T S(x) \left[\frac{\partial A}{\partial x} \right] v dx \\ &= \int_0^l \left[\frac{\partial v^T}{\partial x} \right] S(x) A v dx \end{aligned} \quad (7-12)$$

Using the same discussion about conjugate of operator $[SA]$ the following can be derived.

$$\int_0^l v^T S(x) A \frac{\partial v}{\partial x} dx = \int_0^l \left[[S(x) A]^T v \right]^T \frac{\partial v}{\partial x} dx \quad (7-13)$$

Also,

$$\int_0^l \left[\frac{\partial v^T}{\partial x} \right] S(x) A v \, dx = \int_0^l \left\{ [S(x) A] v \right\}^T \cdot \frac{\partial v}{\partial x} \, dx. \quad (7-14)$$

If $[S(x) A] = [S(x) A]^T$ equation (7-13) becomes identical to (7-14) and hence by substitution into (7-12), one finds

$$2 \int_0^l v^T \cdot [S(x) A] \frac{\partial v}{\partial x} \, dx = \int_0^l \frac{\partial}{\partial x} [v^T S(x) A v] \, dx - \int_0^l v^T \left[\frac{\partial S(x)}{\partial x} \right] A v \, dx - \int_0^l v^T S(x) \left[\frac{\partial A}{\partial x} \right] v \, dx$$

Therefore,

$$\frac{d\mathcal{L}}{dt} = - v^T S(x) A v \Big|_0^l + \int_0^l v^T \left\{ \frac{\partial S}{\partial x} A + \frac{S \partial A}{\partial x} - 2 S(x) B \right\} v \, dx \quad (7-15)$$

Based on the resulting equation (7-15), the following conditions must be satisfied for guaranteeing $\frac{d\mathcal{L}}{dt} \leq 0$

a) Boundary constraint:

$$- v^T(t, l) S(l) A(l) v(t, l) + v^T(t, 0) S(0) A(0) v(t, 0) \leq 0 \quad (7-16)$$

b) Interior constraint:

$$\left\{ \frac{\partial S}{\partial x} A + S(x) \frac{\partial A}{\partial x} - 2 S(x) B \right\} \text{ must be negative definite for } x \in (0, l) \quad (7-17)$$

The operator A in perturbed linear model of MPD as derived in equations (7-6) to (7-9) has already been put into symmetric form. Therefore it is sufficient to find a symmetric matrix function $S(x)$ with discussed properties such that (7-16) and (7-17) are satisfied.

It should be mentioned that the perturbations from equilibrium states which are governed by (7-6) to (7-9) are subject to the following initial and boundary conditions

$$v(0,x) = v_0(x) \quad , \quad v(t,0) = 0 \quad (7-18)$$

where

$$v_0(x) \in L^2(0,\ell)$$

These conditions represent a situation where the plasma flows into the thruster with affixed state (i.e., invariant with time). Due to the perturbation in the control inputs, there is an initial perturbation inside the thruster which is represented by $V_0(x)$ in (7-18). Considering $S(x)$ an operator even simpler than what is required, i.e.,

$$S(x) = \begin{bmatrix} q(x) & 0 & 0 \\ 0 & q(x) & 0 \\ 0 & 0 & q(x) \end{bmatrix} = q(x) I$$

where $q(x)$ is a scalar function, then for $S(x)$ to be positive definite it is sufficient to have, $q(x) > 0$ for $x \in [0,\ell]$. Therefore, the inequality in condition (7-16) would become

$$-q(\ell) V^T(t,\ell) A(\ell) v(t,\ell) + 0 \leq 0$$

where $v(t,0) = 0$.

It can be seen that $A(\ell)$ must be positive definite.

$$A(\ell) = A(x) \Big|_{x=\ell} = \begin{bmatrix} u_e & \sqrt{RT_e} & 0 \\ \sqrt{RT_e} & u_e & \sqrt{(\gamma-1)RT_e} \\ 0 & \sqrt{(\gamma-1)RT_e} & u_e \end{bmatrix}_{x=\ell}$$

The eigenvalues of A are:

$$\lambda_1 = u_e \quad , \quad \lambda_2 = u_e + \sqrt{\gamma RT_e} \quad , \quad \lambda_3 = u_e - \sqrt{\gamma RT_e}$$

At $x = \ell$ for accelerating flow it is practically desirable to have $M > 1$ or $u_e > \sqrt{\gamma RT_e}$.

Hence in this case $A(\epsilon)$ will be positive definite and condition (7-16) is satisfied. To evaluate the requirement for condition (7-17), one has to find $\frac{\partial S}{\partial x}$ and $\frac{\partial A}{\partial x}$

$$\frac{\partial S}{\partial x} = S' = q'(x) I$$

$$\frac{\partial A}{\partial x} = A' = \begin{bmatrix} u'_e & \frac{1}{2}\sqrt{RT_e} \frac{T'_e}{T_e} & 0 \\ \frac{1}{2}\sqrt{RT_e} \frac{T'_e}{T_e} & u'_e & \frac{1}{2}\sqrt{(\gamma-1)RT_e} \frac{T'_e}{T_e} \\ 0 & \frac{1}{2}\sqrt{(\gamma-1)RT_e} \frac{T'_e}{T_e} & u'_e \end{bmatrix}$$

$$S'(x) A(x) + S(x) A'(x) - 2S(x) B(x) = q(x) \left\{ \frac{q'(x)}{q(x)} A(x) + A'(x) - 2 B(x) \right\}$$

Since $q(x) \neq 0$ one should have the following condition:

$$\frac{q'(x)}{q(x)} = Q(x)$$

$$M_L = [Q(x) A(x) + A'(x) - 2 B(x)] \quad (7-18)$$

where M_L must be negative definite.

$$M_L = \begin{bmatrix} u_e \left[Q + \frac{u'_e}{u_e} \right] & a_T \left[Q + 2 \frac{u'_e}{u_e} \right] \\ a_T \left[Q + \frac{3}{2} \frac{T'_e}{T_e} + 2 \frac{u_e u'_e}{RT_e} + 2 \frac{u'_e}{u_e} \right] & u_e \left[Q + \frac{u'_e}{u_e} + \frac{T'_e}{T_e} \right] \\ u_e \left[\frac{2T'_e}{(\gamma-1)T_e} + \frac{2\sqrt{(\gamma-1)u'_e}}{u_e} \right] & -\sqrt{(\gamma-1)} a_T \left[Q + \frac{1}{2} \frac{T'_e}{T_e} + \frac{2 \left[\frac{U_1}{\rho_e c_V} + T'_e \right]}{(\gamma-1)T_e} \right] \\ 0 & \\ -\sqrt{(\gamma-1)} a_T \left[Q + \frac{3}{2} \frac{T'_e}{T_e} + \frac{2u'_e}{u_e} \right] & \\ u_e \left[Q + \frac{u'_e}{u_e} + \frac{2T'_e}{T_e} \right] & \end{bmatrix}$$

where $a_T = \sqrt{RT_e}$.

One way to find an apriori requirement on the system such that stability of perturbation is guaranteed, i.e., condition (7-18) be satisfied, is to let $Q(x)$ be negative and

$$|Q(x)| \gg \max f$$

where

$$f = \left| 3/2 \frac{T'_e}{T_e} \right| + \left| \frac{2u'_e}{u_e} \right| + 2 \left| \frac{u_e u'_e}{RT_e} \right| + 2 \left| \frac{U_1}{\rho_e RT_e} \right|$$

hence

$$M_L \equiv \begin{bmatrix} u_e Q & a_T Q & 0 \\ a_T Q & u_e Q & \sqrt{\gamma-1} a_T Q \\ 0 & \sqrt{\gamma-1} a_T Q & u_e Q \end{bmatrix}$$

Then condition (7-18) would become

$$\frac{M_L}{Q} = \begin{bmatrix} u_e & a_T & 0 \\ a_T & u_e & \sqrt{\gamma-1} a_T \\ 0 & a_T & u_e \end{bmatrix} \text{ is positive definite}$$

Again,

$$\lambda_1 = u_e, \quad \lambda_2 = u_e + \sqrt{\gamma RT_e}, \quad \lambda_3 = u_e - \sqrt{\gamma RT_e},$$

for $\frac{M_L}{Q}$ to be positive definite, one should have $\lambda_3 > 0$

or,

$$u_e > \sqrt{\gamma RT_e}, \quad M > 1 \quad \text{for } x \in (0,1)$$

This restates the situation that flow is supersonic through the interior of the MPD thruster.

Therefore one can conclude that a sufficient condition for the stability of the system with respect to initial perturbations is that the flow regime remains supersonic throughout the thruster.

8. FUTURE WORK

The analysis of Section 7 indicates that in the case of supersonic thrusters there was sufficient conditions for system stability after an initial perturbation. However, in general, one would like to have the system stabilized for all flow patterns (all steady state solutions), which are of interest as discussed in Section 6. In order to address this problem the inherent characteristics of the system should be analyzed. Such analysis often involves information about characteristics of the system of partial differential equations.

Recalling the system of equations (7-7) for a general case

$$\frac{\partial v}{\partial t} + A \frac{\partial v}{\partial x} + Bv = 0. \quad (8-1)$$

Where $A(x)$ is symmetric and has eigenvalues of the general form;
 $\lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_p(x) < 0 < \lambda_{p+1}(x) \leq \dots \leq \lambda_n(x)$.

The eigenvalues show the directions of the characteristic lines of PDE. Also there exists a continuously differentiable matrix $O(x)$, based on the system eigenvectors such that $O^{-1}(x) A(x) O(x) = \Lambda(x)$. If one considers a new set of states w such that

$$v = O(x)w$$

then substitution of v into (8-1) results in

$$O(x) \frac{\partial w}{\partial t} + A(x) O(x) \frac{\partial w}{\partial x} + \left[A(x) \frac{\partial O(x)}{\partial x} + B O(x) \right] w = 0$$

or

$$\frac{\partial w}{\partial t} + O^{-1}(x) A(x) O(x) \frac{\partial w}{\partial x} + O^{-1} \left[A(x) \frac{\partial O(x)}{\partial x} + B O(x) \right] w = 0$$

Therefore one can conclude;

$$\frac{\partial w}{\partial t} + \Lambda(x) \frac{\partial w}{\partial x} + \beta(x)w = 0 \quad (8-2)$$

Since $\Lambda(x)$ is a diagonal matrix of the system eigenvalues and from the fact that λ_1 to λ_p are negative and λ_{p+1} to λ_n are positive, then Λ can be decomposed as follows.

$$\Lambda = \begin{bmatrix} \Lambda^- & 0 \\ 0 & \Lambda^+ \end{bmatrix}, \quad \Lambda^- = \text{diag}(\lambda_1, \dots, \lambda_p) \\ \Lambda^+ = \text{diag}(\lambda_{p+1}, \dots, \lambda_n) \quad (8-3)$$

The corresponding decomposition can be applied to the states

$$w = \begin{bmatrix} w^- \\ w^+ \end{bmatrix} \quad (8-4)$$

Therefore (8-2) can be reduced to a set of ordinary differential equations of the following form.

$$\frac{dw_k}{dt} = \frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = \frac{\partial w_k}{\partial t} + \left[\frac{dx}{dt} \right]_k \frac{\partial w_k}{\partial x} = -\beta_k(x)w \\ k = 1, 2, \dots, n \quad (8-5)$$

where w_k is k th element of the vector valued function w . Moreover, λ_k represents the direction of the k th characteristic line, hence $\frac{d(\cdot)}{dt}$ is the directional derivative along the corresponding characteristic line. On the right hand side of (8-5) β_k is the k th row of the matrix valued function $\beta(x)$. The set of n ordinary differential equations (8-5), are coupled by the term $\beta_k(x)w$. Initial values of $t = 0$ can be given as

$$w(x,0) = w_0 \in (L_2(0,\ell); E^n)$$

Considering the boundary conditions, at a point on the boundary $x = 0, t = t_0$ as shown in figure (8-1), one finds that the characteristics with negative and positive eigenvalues arrive at that point with negative and positive slopes respectively. The "incoming information" consists of values of w_k associated with characteristic line $C_k(0, t_0)$ with negative slope for $k = 1, \dots, p$. The "outgoing information" consists of values of w_k associated with positive slope characteristic $C_k(0, t_0)$, $k = p+1, \dots, n$. Hence along the boundary $x = 0$ the values of w^+ should be known. It is clear that

along boundary $x = 1$, the orientation of characteristics will be reversed and hence the values of w^- should be known.

At this stage one can impose the question of stabilizability of linear hyperbolic systems, namely those systems such as MPD thruster with an initial perturbation. The stabilizability question would be as such: what control inputs should be used to bring the system to a stable state? This question can be approached in two different ways. The first approach is the idea of null controllability which is more restrictive and considers determination of a control (applied on the boundary) such that for a given initial perturbation w_0 , after time $t > 0$,

$$w(x, T) = 0.$$

It can be seen that the required time T should satisfy the following inequality

$$T \geq -\int_0^{\ell} \frac{dx}{\lambda_p(x)} + \int_0^{\ell} \frac{dx}{\lambda_{p+1}(x)}$$

By this approach the time-space plane (domain of the system) can be divided into three separate regions as shown in figure (8-2). After finding the values of states in each region, the value of $w(\ell, t)$ will be evaluated. Since

$$w(\ell, t) = \begin{bmatrix} w^-(\ell, t) \\ w^+(\ell, t) \end{bmatrix},$$

then knowledge of both incoming and outgoing information along $x = \ell$ would lead to the determination of a boundary value control at this boundary. 20

The second approach is more or less based on the similar ideas as those of Section 7. In this approach an energy function (i.e. Lyapunov function) is defined. Depending on the type of system, two types of controllers can be sought. These are "boundary" controller or "body force" controller. For a wave equation (linear and nonlinear) it is shown [15] that a control force at the boundary can steer the system to a stabilized

state. Whereas in the case of a linear wave equation with a "body force" (i.e. internal controller) it is shown [16] that the system can be controlled to a "zero energy" state.

In both examples the energy functions were proposed as:

$$E = \frac{1}{2} \int_x \left(\frac{\partial u}{\partial x} \right)^2 + C^2 \left(\frac{\partial u}{\partial x} \right)^2 dx .$$

Although both of these controllability approaches have been applied to much simpler systems, it is desirable to answer some of these controllability questions in reference to the MPD system via these approaches in the future research.

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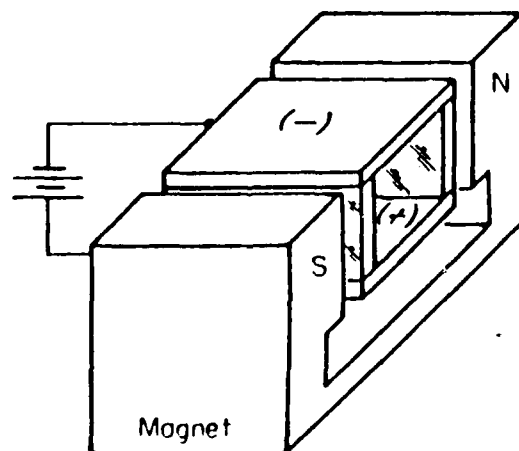
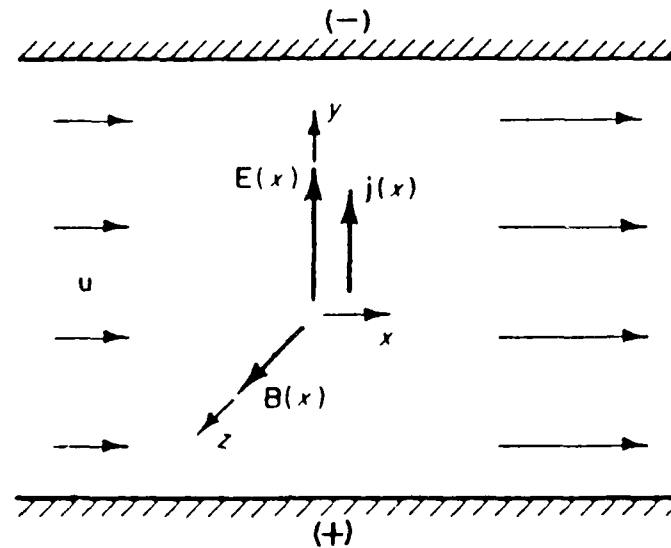


Figure 5-1. Schematic of configuration of flow and fields for one-dimensional electromagnetic steady flow accelerator.

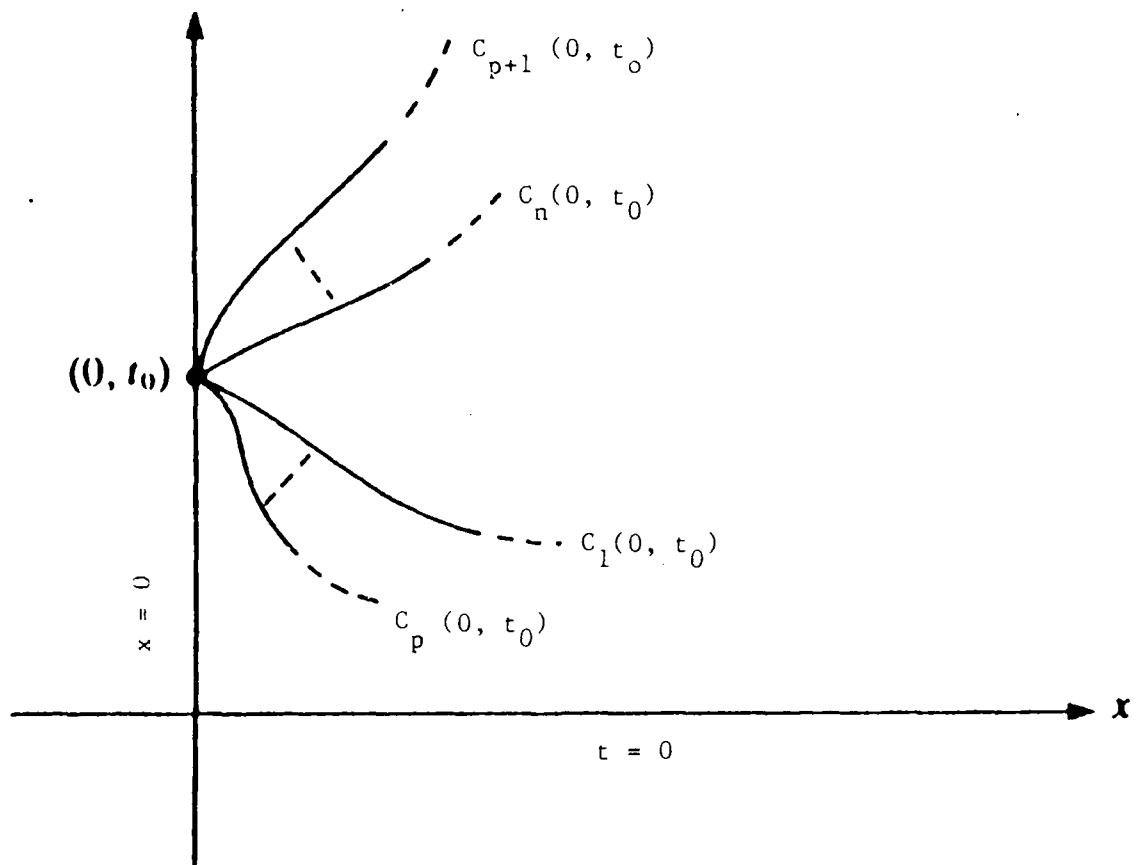


Figure 8-1 Characteristics configuration at $x = 0$ and $t = t_0$.

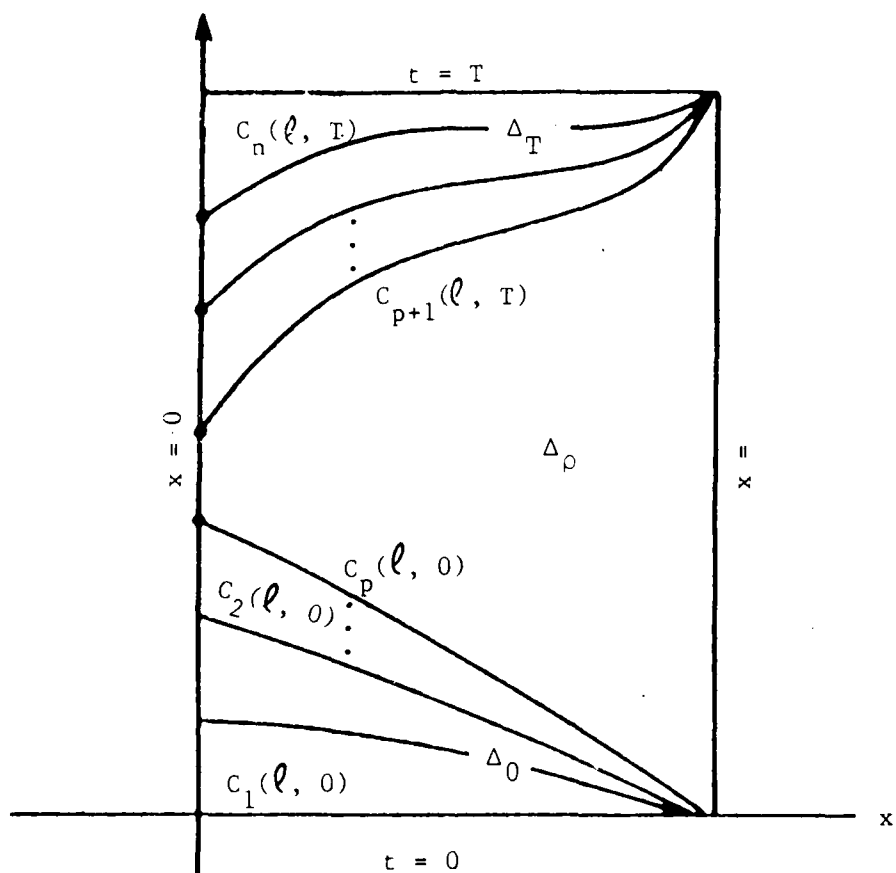


Figure 8-2 Schematic of decomposition of t - x plane into 3 regions based on characteristics.

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ABSTRACT

A magneto-plasma-dynamic thruster is subject to a number of instabilities, some of them associated with ionization or electro-thermal instability. An attempt has been made (Part I) to establish the conditions under which imbalances in power input and electron diffusion can lead to instability. In Part II a new approach to stability of MPD engines is presented. This approach is based on extension of the Lyapunov direct stability theorem to distributed parameter systems. Problem formulation in an operator form is presented. The Lyapunov function is extended to a Lyapunov functional which is an integration over space. This approach has been applied to a magneto-gas dynamic problem, namely a simplified form of an MPD engine. The stability results are presented and discussed. Wave characteristics in both transverse and longitudinal modes are discussed, and upper bounds for wave speeds based on Alfven speed are derived.

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PART I

S.N.P. Murthy

Background

A MPD thruster, shown schematically in Figure 1, may be divided into three regions: (1) initial ionization region with heating of electrons, rapid dissociation and ionization of heavy particles and electron pressure large compared to the partial pressure of heavy particles; (2) uniform arc region with electron temperature nearly in equilibrium, heavy particles being heated by transfer of energy from electrons undergoing Joule heating and electron temperature and current density nearly constant; and (3) nonuniform arc region with electron properties varying appreciably while the advection velocity is transonic or greater.

One important factor in region (2) is that the Joule heating undergoes a change when the plasma current attains a constant value in space. Thus, initially, Joule heating can be expressed by the relation, namely

$$J_H(T_e) \propto I_p^2 T_e^{-3/2} \quad (1)$$

while, when the current attains the constant value, it becomes the following.

$$J_H(T_e) \propto V^2 T_e^{3/2} \quad (2)$$

In Eqs. (1) and (2), I_p , T_e and V represent the plasma current, electron temperature and voltage, respectively. It is important to note that a characteristic diffusion time has to elapse before Joule heating changes from that given by Eq. (1) to that by Eq. (2). At the end of such diffusion time, one can equate J_H in Eqs. (1) and (2) and obtain an expression for the voltage, namely

$$V \propto I_p / T_e^{3/2} \quad (3)$$

Our main objective is to examine the structure of plasma, as given by a set of describing equations, and to determine if there arises a branching and the consequence when such a state of constant current arises.

Describing Equations

The partially ionized gas is considered under the hydrodynamic approximation, given by Refs. 9-10. While the resulting equations are nonlinear partial differential equations, we consider the lowest order approximation to those in terms of the following ordinary differential equations. They will be adequate for illustrating the effects of the plasma current attaining a constant value over a characteristic period of time. The equations of interest are for the number density, temperature and energy balance of relevant species.

$$\frac{dN_e}{dt} \propto S_0 \quad (4)$$

$$\Phi_i \stackrel{\Delta}{=} 0 \quad (10)$$

From Eq. (9), one can solve for T_i and substitute for the Joule heating term from Eq. (1). Similarly Eq. (10) provides another relation between T_i and T_e . It is then of interest to examine the two curves when T_e is small and when it is sufficiently large. After some algebra, the two curves can be utilized to deduce (Ref. 11) the domains in which (dT_e/dt) and (dT_i/dt) are positive or negative, as shown in Fig. 2. The point of intersection of the curves is a stable equilibrium point.

Referring to Eqs. (9) and (10), it is next necessary to substitute for Joule heating from Eq. (2). For $T_e \rightarrow 0$, Φ_e starts at the origin. When T_e becomes finite, the relation between T_i and T_e depends upon the diffusion coefficients S_{De} and S_{Di} . In particular, several solutions can be obtained depending upon the enhancement of diffusion in the gradient zone, that is, depending upon the value of α . Thus, while for small T_e the value of T_i is nearly equal to T_e , at larger values of T_e , T_i decreases and becomes negative, the change depending upon the extent of enhanced diffusion or the value of α . Several illustrative cases can be formulated.

(a) Case of small enhancement of diffusion: Two subcases can be distinguished when α is small, that is, less than a certain value that is a function of V, T_e and N_e . The first subcase is obtained for small input power. Then, as shown in Fig. 3, the system returns to the origin or, for large T_e , a runaway type structure is obtained. The point of intersection of the (dT_e/dt) and (dT_i/dt) curves is then a saddle point.

The second subcase pertains to large input power. Then, there may arise, depending upon the values of system parameters, one of two possibilities: (1) As in Fig. 4, the origin may become unstable while the two trajectories of the solutions do not intersect or (2) as in Fig. 5, there may arise two points of intersection. In the latter case, both the origin and the second point of intersection are unstable while the first point of intersection is a stable node.

(b) Case of large enhancement of diffusion: In the case α is large, the parameters of influence are N_e and the power input. For a given value of power input, there is a value of N_e below which a stable node is obtained as shown in Fig. 6. Above such a value of N_e , the two trajectories do not intersect, as shown in Fig 7. In other words, at any value of N_e there is a maximum amount of power that can be applied, beyond which value of power the enhanced diffusion will drive the system towards the origin.

Discussion

In the region where plasma is being supplied with energy for the heating of ions through the Joule heating of electrons, a branching in the structure of plasma has been shown to arise when the current across the plasma becomes constant and remains at that value over a characteristic period of time related to diffusion of electrons. The diffusion becomes enhanced in the gradient zone. When the net diffusion is small, the various possible states of plasma depend upon the amount of power supplied. On the other hand, when the net diffusion is large, there exists a number density of electrons at each given value of power input above which there appears a disruption in the structure of plasma. This provides a basis for regulating power input in the equilibrium region. The input parameters for such a control system become the electron number density and ion temperature.

Problem Formulation:

Consider the MPD engine shown in Figure 1. The plasma dynamic equations for this engine consist of Maxwell's equations, Ohm's law, conservation of electric charge, equation of state (ideal gas law) and a set of mass, momentum and energy equations [24]. These equations can be written as follows:

Maxwell equations:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \mu \vec{H}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon} \rho_e$$

Ohm's law:

$$J_i = \sigma \left[E_i + \mu_e (\vec{u} \times \vec{H})_i \right] + \rho_e U_i \quad i \triangleq x, y, z$$

Conservation of electric charge:

$$\frac{\partial \rho_e}{\partial t} + \sum_{j=1}^3 \frac{\partial J_j}{\partial x_j} = 0$$

Equation of state (ideal gas law):

$$P = R \rho T$$

Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^3 \frac{\partial}{\partial x_j} (\rho U_j) = 0$$

Conservation of Momentum:

$$\rho \frac{DU_i}{Dt} = - \frac{\partial P}{\partial x_i} + \sum_{j=1}^3 \frac{\partial \tau_{ij}}{\partial x_j} + F_{e_i} + F_{g_i}$$

$$\vec{E} = i E_x(x,t) + j E_y(x,t) + k E_z(x,t)$$

$$\vec{H} = \vec{H}_0 + \vec{h}(x,t)$$

$$= i [H_x + h_x(x,t)] + j [H_y + h_y(x,t)] + k h_z(x,t)$$

current density J and net electric charge ρ_e are

$$\vec{J} = \vec{J}(x,t), \rho_e = \rho_e(x,t).$$

The one-dimensional assumption results in: $\frac{\partial(\cdot)}{\partial y} = 0, \frac{\partial(\cdot)}{\partial z} = 0,$

Based on the aforementioned treatment of the problem, the following describing equations can be derived.

Maxwell's equations:

$$J_x + \epsilon \frac{\partial E_x}{\partial t} = 0 \quad (1)$$

$$J_y + \epsilon \frac{\partial E_y}{\partial t} = - \frac{\partial h_z}{\partial x} \quad (2)$$

$$J_z + \epsilon \frac{\partial E_z}{\partial t} = \frac{\partial h_y}{\partial x} \quad (3)$$

$$\mu_e \frac{\partial h_x}{\partial t} = 0 \quad (4)$$

$$\mu_e \frac{\partial h_y}{\partial t} = \frac{\partial E_z}{\partial x} \quad (5)$$

$$\mu_e \frac{\partial h_z}{\partial t} = - \frac{\partial E_y}{\partial x} \quad (6)$$

Generalized Ohm's law:

$$J_x = \sigma (E_x - \mu_e w H_y) + \rho_e u \quad (7)$$

$$J_y = \sigma (E_y + \mu_e w H_x) + \rho_e v \quad (8)$$

$$J_z = \sigma (E_z + \mu_e u H_y - \mu_e v H_x) + \rho_e w \quad (9)$$

Conservation of electric charge:

$$\frac{\partial \rho_e}{\partial t} + \frac{\partial J_x}{\partial x} = 0 \quad (10)$$

$$\frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} + H_x \frac{\partial w}{\partial x} \quad (17)$$

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial x^2} + \frac{V_x^2}{H_x} \frac{\partial h_z}{\partial x} \quad (18)$$

where $\nu_H = \frac{1}{\sigma \mu_e}$ and $V_x = \sqrt{\frac{\mu_e}{\rho_0}} H_x$. The parameter V_x is defined as x-component of speed of Alfven wave.

(ii.) Longitudinal Mode

The state equation for this mode can be reduced to

$$\frac{\partial h_y}{\partial t} = \nu_H \frac{\partial^2 h_y}{\partial x^2} - H_y \frac{\partial u}{\partial x} + H_x \frac{\partial v}{\partial x} \quad (19)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} + \frac{V_x^2}{H_x} \frac{\partial h_y}{\partial x} \quad (20)$$

$$\frac{\partial u}{\partial t} = -RT_0 \left[\frac{\partial \rho''}{\partial x} + \frac{\partial T''}{\partial x} \right] + \frac{4}{3} \nu \frac{\partial^2 u}{\partial x^2} - \frac{V_y^2}{H_y} \frac{\partial h_y}{\partial x} \quad (21)$$

$$\frac{\partial \rho''}{\partial t} = - \frac{\partial u}{\partial x} \quad (22)$$

$$\frac{\partial T''}{\partial t} = \frac{K}{\rho_0 c_\nu} \frac{\partial^2 T''}{\partial x^2} - \frac{R}{c_\nu} \frac{\partial u}{\partial x} \quad (23)$$

where $\rho'' = \frac{\rho'}{\rho_0}$, $T'' = \frac{T'}{T_0}$, $V_y = \sqrt{\frac{\mu_e}{\rho_0}} H_y$. The parameter V_y is defined as y-component of the speed of Alfven wave.

Lyapunov Functional and Stability Analysis

In this section the Lyapunov Functional approach is applied to each mode of the plasma dynamics. The stability results are derived and discussed.

(i.) For the transverse mode the general operator form of the equation (17) and (18) evolution equation would be

$$\det \begin{bmatrix} \frac{\dot{T}_b}{T_b} - \nu_H \frac{X_b''}{X_b} & H_x \frac{X_w}{X_w} \\ \frac{V_x^2}{H_x} \frac{\lambda_b}{X_b} & \frac{\dot{T}_w}{T_w} - \nu \frac{X_w''}{X_w} \end{bmatrix} = 0 \quad (25)$$

In order to have T and X functions independent from x and t, respectively, it is required that $\frac{\dot{T}}{T} \neq f(x,t)$, $\frac{X''}{X} \neq f(x,t)$. From $\frac{\dot{T}}{T}, \frac{X'}{X} = \text{const.}$ and $\frac{X''}{X} = \text{const}$ it is conclusive to represent X functions in terms of a real periodic function

$$X = A_x e^{i\lambda x/\ell} + A_x^* e^{-i\lambda x/\ell}$$

where $A^* = \text{complex conjugate of } A$

$$\frac{X''}{X} = -\frac{\lambda^2}{\ell^2}, \quad \frac{X'}{X} = \frac{i\lambda}{\ell}$$

and $T = A_T e^{st} \rightarrow \frac{\dot{T}}{T} = S$. From the boundary condition it appears

$$X = A_x \sin \frac{\lambda_n}{\ell}$$

$$\lambda_n = n\pi \quad n = \pm 1, \pm 2, \dots$$

Therefore, equation (25) can be reduced to

$$\det \begin{bmatrix} \left(S + \nu_H \frac{\lambda_n^2}{\ell^2} \right) & -H_x i \frac{\lambda_n}{\ell} \\ -\frac{V_x^2}{H_x} i \frac{\lambda_n}{\ell} & S + \nu \frac{\lambda_n^2}{\ell^2} \end{bmatrix} = 0$$

The point spectrum of operator A, i.e., $\sigma_p(A)$ can be found as $\sigma_p(A) = \{S \in \sigma(A) \mid (SI-A) \text{ is not one to one} \}$

$$S^2 + S(\nu_H + \nu) \frac{\lambda_n^2}{\ell^2} + V_x^2 \frac{\lambda_n^2}{\ell^2} + \nu V_H \frac{\lambda_n^4}{\ell^4} = 0$$

The roots of this equation are

$$\int_0^\ell \left(\frac{\partial f}{\partial x} \right)^2 dx \geq \pi^2 \int_0^\ell f^2 dx$$

and

$$\int_0^\ell \left[h_z \frac{\partial w}{\partial x} + w \frac{\partial h_z}{\partial x} \right] dx = h_z w \Big|_0^\ell = 0$$

the following can be resulted.

$$\begin{aligned} \dot{V} &= -2 \int_0^\ell \left[\alpha_1 \nu_H \left(\frac{\partial h_z}{\partial x} \right)^2 + \alpha_2 \nu \left(\frac{\partial w}{\partial x} \right)^2 \right] dx \\ &\leq -2\pi^2 \int_0^\ell (\alpha_1 \nu_H h_z^2 + \alpha_2 \nu w^2) dx \end{aligned}$$

if $\alpha_2 = 1$, $\alpha_1 = V_x^2/H_x^2$ then

$$V = \frac{V_x^2}{H_x^2} \|h_z\|^2 + \|w\|^2$$

$$\dot{V} \leq -2\pi^2 \left[\frac{V_x^2}{H_x^2} \nu_H \|h_z\| + \nu \|w\|^2 \right] \quad (28)$$

In this case (28) shows that \dot{V} is negative definite and that proves the stability of the transverse mode using Lyapunov approach.

(ii.) For longitudinal mode of wave propagation the generic form of evolution equation (24) is considered

$$\underline{Z} = \begin{bmatrix} h_y \\ v \\ u \\ \rho'' \\ T'' \end{bmatrix} \in L^2(0, \ell) \quad \text{and} \quad \underline{Z}(0, t) = \underline{Z}(\ell, t) = 0.$$

In this case \underline{A} in equation (24) is a linear operator with the domain in a separable Hilbert space of state function which maps \underline{Z} onto itself. From equations (19) to (23) \underline{A} is formed as,

In general S_n can be written as a combination of real and imaginary parts.

$$S_n = \text{Re}(S_n) + i \text{Im}(S_n) .$$

This resulted in

$$\omega_n^2 = V_x^2 \frac{\lambda_n^2}{\ell^2} - \frac{(\nu_H - \nu)^2}{4} \frac{\lambda_n^4}{\ell^4}$$

Obviously, in order to have a wave (under damped condition) ω_n^2 has to be positive. Otherwise, ($\omega_n^2 \leq 0$) there is no wave propagation in the transverse mode. Hence, wave speed can be defined as

$$\hat{V}^2 = \frac{\ell^2 \omega_n^2}{\lambda_n^2} = V_x^2 - \frac{(\nu_H - \nu)^2}{4} \frac{\lambda_n^2}{\ell^2}$$

or

$$\frac{\hat{V}^2}{V_x^2} + \frac{(\nu_H - \nu)^2}{\frac{4\ell^2}{\lambda_n^2} V_x^2} = 1$$

which has an elliptical shape, as shown in Figure 8. The maximum of the transverse wave speed is limited by Alfvén speed in the x-direction in the case of $\nu_H = \nu$. Moreover, for the transverse wave to exist, the maximum difference in $(\nu_H - \nu)$ is caused by minimum λ_n (i.e. $\lambda_{n,\min} = \pi$). Therefore, if

$$|\nu_H - \nu| \leq \frac{2\ell V_x}{\pi} ,$$

the generation of transverse wave is plausible.

For the longitudinal waves, due to the complexity of characteristic equation the case that $\nu = K = 0$ and $\nu_H \neq 0$ is considered. The characteristic equation, then will be,

$$S^4 + \lambda^2 S^2 [\gamma R T_o + (V_x^2 + V_y^2)] + \gamma R T_o V_x^2 + \nu_H \{S^3 \lambda^2 + S \lambda^4 \gamma R T_o\} = 0$$

In complex $\frac{S^2}{\lambda}$ plane for $\nu_H = 0$ the following can be obtained.

$$S^4 + \lambda^2 S^2 [\gamma R T_o + V^2] + \gamma R T_o V_x^2 = 0$$

$$\frac{S_{1,2}^2}{\lambda^2} = \frac{1}{2} \left[-(\gamma R T_o + V^2) \pm \sqrt{(\gamma R T_o + V^2)^2 - 4\gamma R T_o V_x^2} \right]$$

where $V^2 = V_x^2 + V_y^2$. For the case of ν_H approaching infinity S_3 is zero and

u, v, w: velocity vector components in x,y and z directions
 x, y, z: Spatial coordinates
 X_j : Spatial coordinates for $j = 1, 2, 3$

Greeks

ϵ : Dielectric constant
 μ, ν Viscosity, kinematic viscosity
 μ_e : Magnetic permeability
 ρ, ρ' : Plasma density, perturbation in density
 ρ_e : Excess electric charge/volume
 σ : Electrical conductivity
 ν_H : $\frac{1}{\sigma\mu_e}$

subscripts

x, y, z: Variable subscribed in the direction of x, y and z.

Notation

T_h, T_w : Time dependent part of variable in subscript
 X_h, X_w : x dependent part of variable in subscript.

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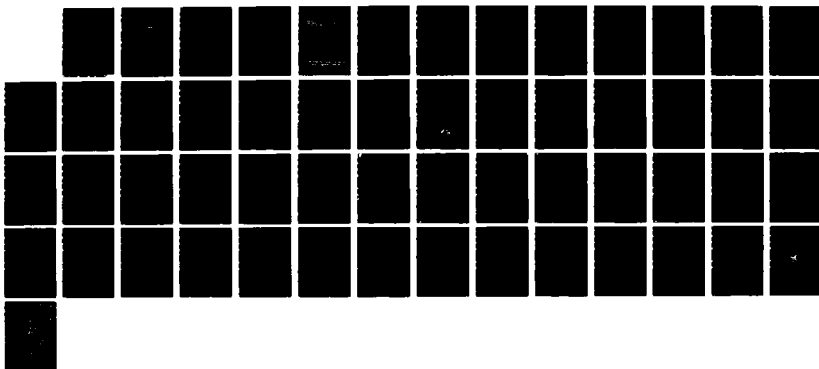
DIAGNOSTICS FOR INTELLIGENT CONTROL OF MPD
(MAGNETO-PLASMA DYNAMIC) ENGIN (U) PURDUE UNIV
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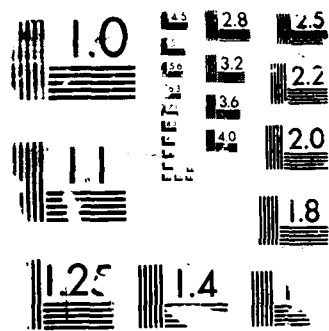
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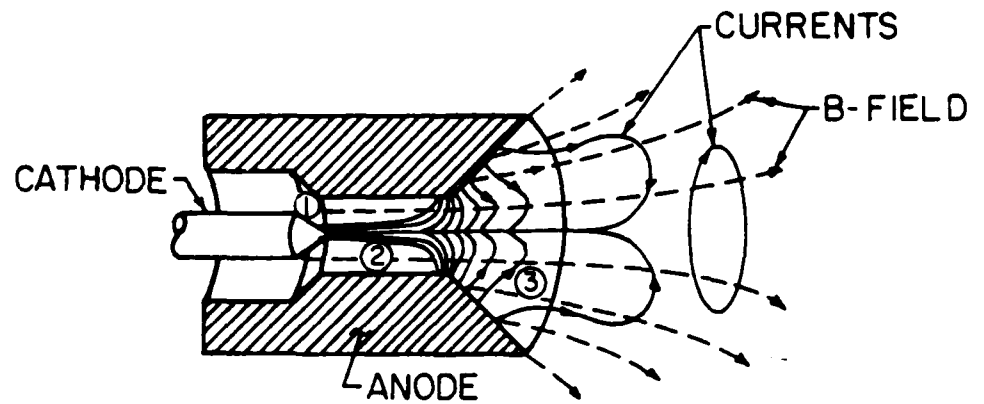


FIGURE 1. MPD ARC THRUSTER

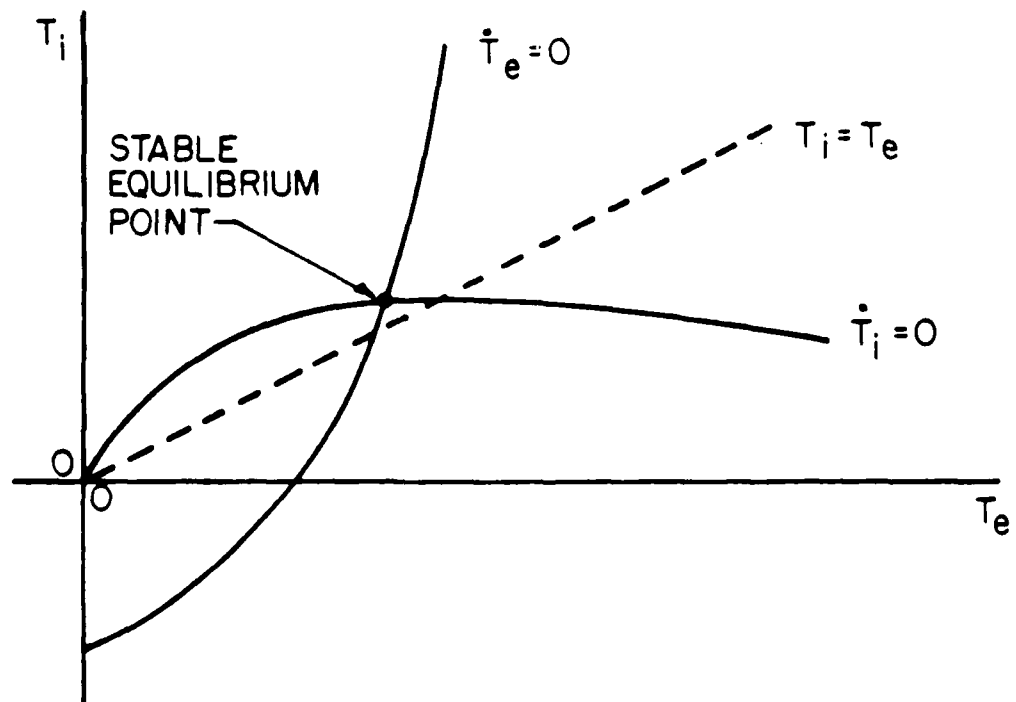


FIGURE 2.

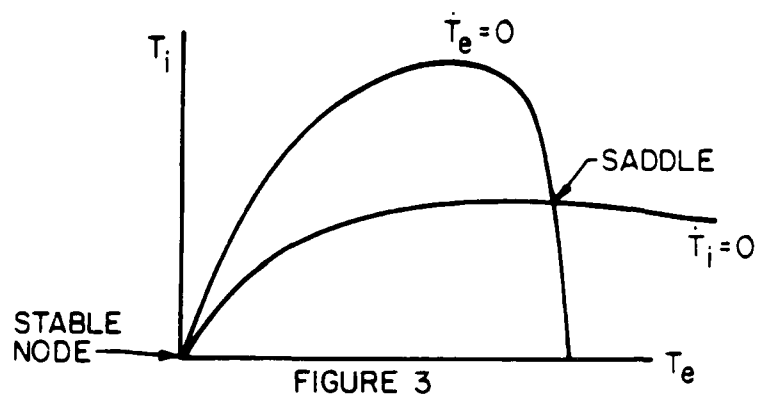


FIGURE 3

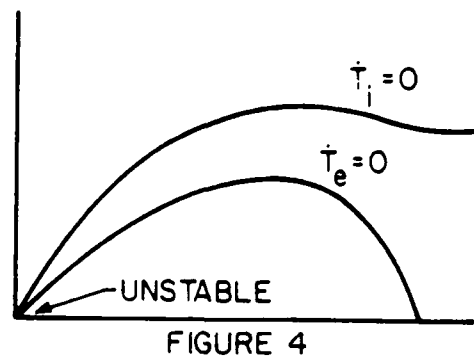


FIGURE 4

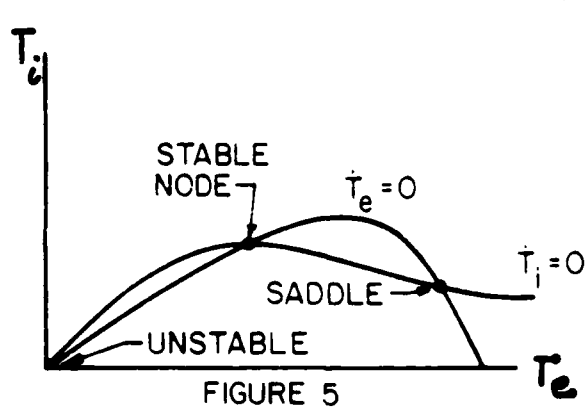


FIGURE 5

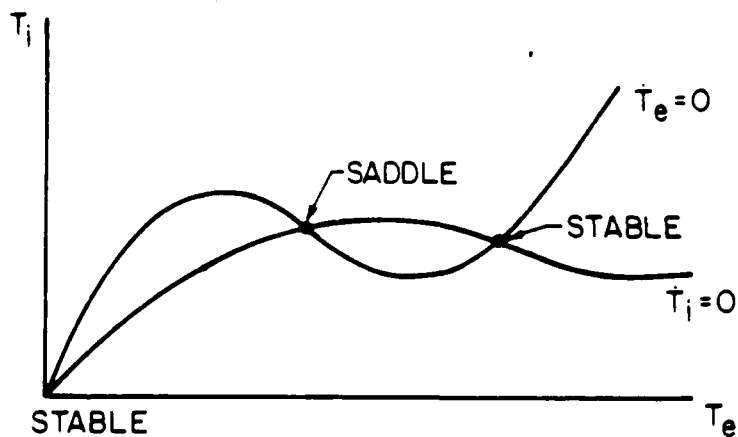


FIGURE 6

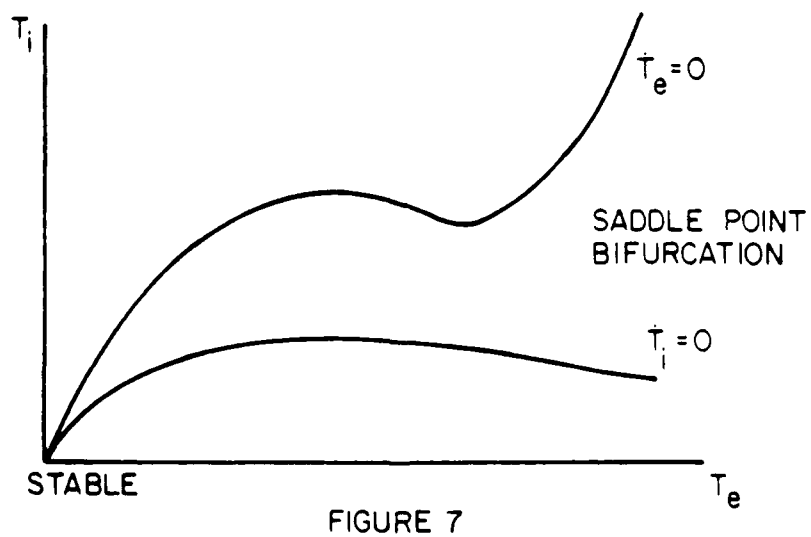


FIGURE 7

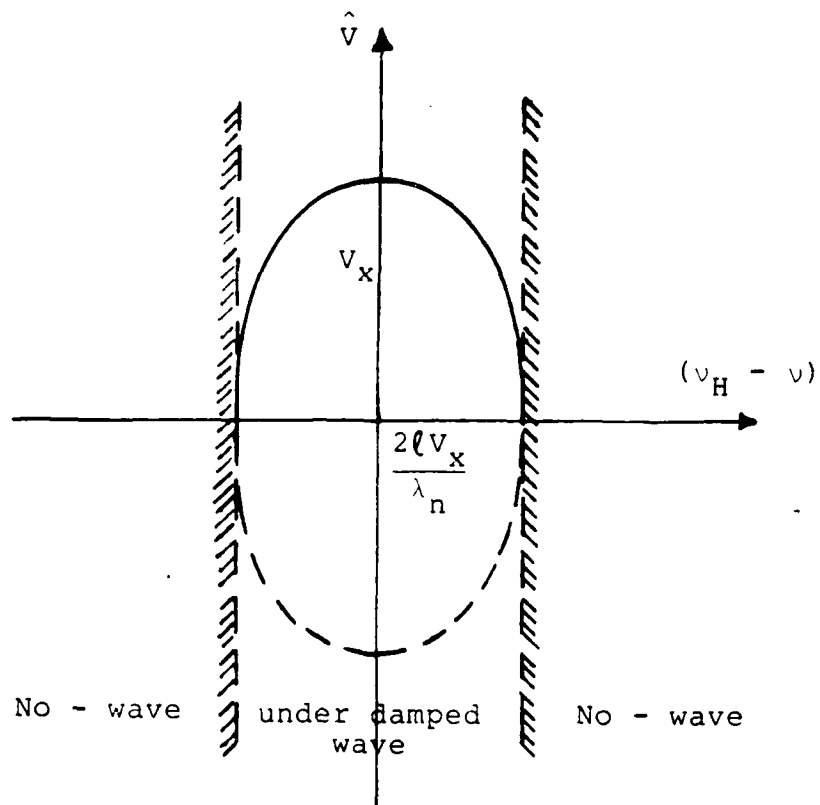


Figure 8: Wave Speed for Transverse Mode

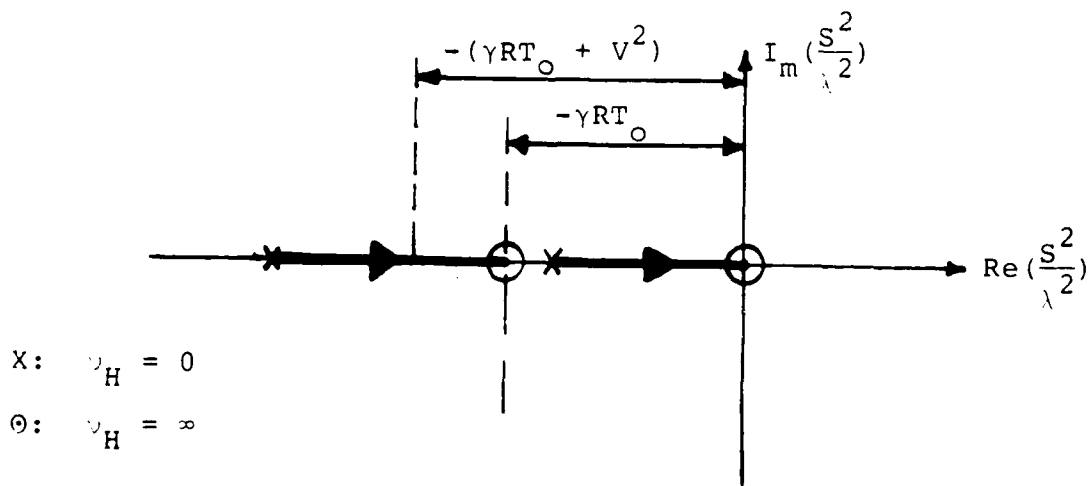


Figure 9: Eigenvalues for Longitudinal Mode

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**Observability of Heat Transfer to MPD
Thruster Electrodes**

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OBSERVABILITY OF HEAT TRANSFER TO MPD THRUSTER ELECTRODES*

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Abstract

An attempt has been made to set up a model for heat transfer to the cathode of a MPD thruster and to conduct an observability analysis of a model energy balance equation. Considering a monoatomic gas that is singly ionized and is in a state of nonequilibrium with different species at different temperatures, the energy balance in the vicinity of the electrode is described in terms of seven regions including a "mushy" region involving phase change processes. The observability analysis is conducted on a model energy balance equation applicable to one of the regions. Since a MPD engine is a nonlinear distributed parameter system, the concept of observability analysis for such systems is derived and presented. An observability criteria for finite and infinite dimensional systems, based on the transformation of nonlinear systems to observable canonical form, is studied. A less restrictive approach for observability of distributed parameter systems is presented. This technique is applied to two cases of available measurements of the MPD engine. The resulting conditions on the control inputs to the engine are presented.

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INTRODUCTION

The electrode region of a magneto-plasma-dynamic (MPD) engine continues to present several complexities both in analysis and measurements that can serve to explain phenomenology and to permit predictions of such quantities as heat transfer and wear [1-6]. The complexities are further accentuated when the operating conditions are close to the occurrence of onset instability and other critical regimes [7-9]. The electrode processes are of interest both during the start-up phase of continuously operated systems and in pulsed devices. In all cases attention has to be paid to the state of the plasma and the processes that are dominant in different zones as the electrode surface is approached from the free stream condition. In general, the plasma may be multi-component, multi-temperature, including elastic and inelastic collisions, with complex reaction rates, subjected to Joule heating, and acted on by induced and applied magnetic fields. Furthermore, one has to account for electrode surface reactions and resulting changes both in the plasma and the mechanical structure of the electrode itself. A number of zones can be identified in the vicinity of the electrode based on characteristic length scales such as the recombination length (ℓ_R), the molecular mean free path ($\ell_{a,e,i}$, a, e and i referring to atom, electron and ion, respectively) and the Debye length (ℓ_D). The rate-governed processes in those zones will depend further upon characteristic times.

A beginning has been made in Reference [10] on the analysis of a plasma boundary layer under the assumption of an equilibrium plasma in the absence of Joule heating and any ambient magnetic field. Although the boundary layer pertains to a power generator (and not a plasma generation device), important findings have been made on the relative importance of inertia of electrons and ions and of charge separation in the vicinity of electrodes under various levels of current input. The energy balance equation is obviously left out of account in the analysis. On the other hand, inclusion of considerations of nonequilibrium and Joule heating make it imperative to include the energy balance equation in the analysis.

The analysis of a complex flowfield such as that of a MPD engine can be undertaken from several points of view: prediction of performance, analysis for distinguishing and ordering the importance of various processes and determination of stability, optimality, observability and controllability of the system represented by a set of describing equations. The latter has been the basis of recent investigations undertaken at Purdue University.

The objectives of the current work in that context are two: (1) To obtain a reasonably detailed set of describing equations for the vicinity of the electrode, including Joule heating and energy balance, and (2) to present the concept of observability for such nonlinear systems as MPD engines, where system parameters change with both time and space (Distributed Parameter System - DPS).

In general, distributed parameter systems are represented by partial differential equations, relating informations on spatial domain of the system to the time domain. These systems can be considered infinite dimensional as opposed to finite dimensional notion for lumped parameter systems.

Control of the DPS's are usually achieved by a parameter approximation of the system. This approach, using finite element or finite difference technique has been

useful for control design of some systems. In general, intrinsic properties of DPS can not be predicted. The stability, controllability and observability considerations of such systems often can not be achieved with discretized models. The formulation and treatment of continuous model approach from the standpoint of control theory has been investigated [11,12].

Many researchers have studied the stability of linear DPS [13,14]. One of the first attempts using Lyapunov's direct method for dimensional systems was made by [15]. Based on an abstract theory of Lyapunov's stability scheme, references [16,17] have defined Lyapunov's functions on Hilbert and Banach spaces. An application of this approach has been studied for magneto-plasma-dynamic (MPD) system modeled as a DPS [18].

The controllability of linear DPS has been formulated and studied by [12,14,19]. Also the researchers [20] have investigated and derived a controllability criteria for special classes of DPS. Reference [21] is attempting to relate the controllability of nonlinear finite dimensional systems originated by [22,23] to a class of nonlinear DPS.

In a distributed parameter system the question of observability may cover two types of information: the criteria which determines whether the measurements contain sufficient conditions to uniquely describe the state of the system; and the location of sensors in order to provide feedback information. This paper is primarily devoted to the first aspect of the problem for the purpose of establishing a criteria for the observability of DPS. One of the initial investigations in this area, Wang [22], has discussed certain aspects of observability for distributed parameter systems and has given a definition with respect to initial state recovery of systems. Goodson and Klein [24] define observability as the ability to establish the uniqueness of a solution of the system. Other investigators [25-29] have studied the observability of systems which belong to the classes of linear distributed parameter systems. For such systems, often, it is necessary and sufficient to have a non-zero inner product of measurement distribution with infinite dimensional eigenvector of the system's differential operator.

For nonlinear systems, in general, the eigenfunction can not be determined and consequently the methods used for linear systems would not be applicable to nonlinear infinite dimensional systems. For nonlinear finite dimensional systems many researchers have tried to transform the nonlinear system equations into a set of linear equations with a specific canonical form. References [30,31] present the conditions on the existence of such transformations. An observable canonical form for nonlinear systems is constructed in [32].

In this study, the objective is to find a region in the state space for which the measurements of the states would lead to reconstruction of the system (i.e., the unique behavior of the system is feasible).

GENERIC MODEL FOR ELECTRODE REGION

Figure 1 presents a schematic of the electrode region of a MPD engine. A thermal boundary layer is postulated that may be different from the momentum boundary layer. The Debye length and the mean free path of species are both small compared to the recombination length but ℓ_D may be either less than or be comparable to $\ell_{e,i}$.

The working fluid is assumed to be a monatomic gas that is undergoing single ionization so that the species considered are atoms, ions and electrons obeying Maxwell-Boltzmann statistics. The plasma in the free stream is assumed to be partially ionized with a possibility of both thermal and charge nonequilibrium. The electron and ion temperatures (T_e, T_i) are therefore assumed to be different. The Saha equation for reaction is modified in order to take into account non-equilibrium and inelastic collisions. The plasma is assumed to be subject to Joule heating and to induced and applied electro-magnetic fields.

The state of plasma within $\ell_{e,i}$ and therefore ℓ_D may only be described in terms of collisionless particles. It may be pointed out that $\ell_{e,i}$ and ℓ_D are not sharply defined boundaries.

Within the electrode, two regions have been identified in Figure 1: the first is a region where melting and evaporation may be taking place and the second, a region with pure conductive heat transfer. The first is referred to as a "mushy" region.

The describing equations applicable to the various regions are provided in Appendix I. At this stage it is not the objective either to match the different regions on an asymptotic basis or to undertake numerical calculations. Hence no boundary conditions are given for the different regions.

ENERGY BALANCE AT A CATHODE

As a particular example of electrode-associated processes, we examine the energy balance in the vicinity of a cathode. The equations describing the state of the plasma including non-equilibrium and multi-temperature effects are presented in Appendix I. Referring to Figure 1, the energy balance equations applicable to different regions are presented in Appendix II along with the associated current flux equation. It is assumed that the cathode material is catalytic and recombination processes within the material give rise to a flux of neutrals into the Debye region and also to heat generation within the electrode. A region, referred to as the "mushy" region, is identified in the electrode material adjoining the plasma. In that region it is assumed that phase change processes, solid to liquid, liquid to vapor and also solid to vapor, are expected to occur.

The energy balance equation at the internal surface of the cathode marked in Figure 1 as $y=0$ may be written as follows.

$$\dot{q}_{\text{solid}} = \dot{q}_p + \dot{q}_{\text{rad}} + \dot{Q}_{\text{sur}} - [\dot{Q}^* + \dot{q}_{\text{cond}} + \dot{q}_{\text{elect}}]_{\text{mushy}}$$

where \dot{q}_p is the net energy transfer to the surface from the plasma due to the kinetic energy and potential energy of the particles, \dot{q}_{rad} , the net radiation to the surface from the plasma, \dot{Q}_{sur} , the energy generation at the surface due to recombination of ions and electrons, \dot{Q}^* , is the energy associated with phase change and, \dot{q}_{cond} , energy conducted into the mushy region.

OBSERVABILITY ANALYSIS

The general form for the set of governing equations of the system under investigation can be expressed as

$$\dot{V} = f(V, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial y^2}, \dots, U) \quad (1)$$

$$y = h(V)$$

where V is the distributed state vector and y is the scalar output (measurement).

A nonlinear transformation $T: V \rightarrow V^*$ with,

$$V = T(V^*) \quad (2)$$

such that system of (1) is transformed into a Brunovsky canonical form [22] given by

$$\begin{aligned} \dot{V}^* &= AV^* + \gamma(y, U) = f^*(V^*, U) \\ y &= CV^* \end{aligned} \quad (3)$$

with

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (4)$$

and

$$C = [0, 0, \dots, 0, 1] \quad (5)$$

Hence

$$\dot{V} = \frac{dT}{dt}(V^*) = \langle \nabla T(V^*) \cdot \frac{dV^*}{dt} \rangle \quad (6)$$

$$f(V, U) = \langle \nabla T(V^*) \cdot f^*(V^*, U) \rangle \quad (7)$$

by differentiation with respect to V_k^* ,

$$\frac{\partial f(V, U)}{\partial V_k^*} = \frac{\partial(\nabla T(V^*))}{\partial V_k^*} \cdot f^* + \nabla T(V^*) \cdot \frac{\partial f^*}{\partial V_k^*} \quad (8)$$

$$\text{L.H.S.} = \frac{\partial f}{\partial V} \frac{\partial T(V^*)}{\partial V_k^*} \quad (8-a)$$

First term of the R.H.S. is

$$\nabla \left(\frac{\partial T}{\partial V_k^*} \right) \cdot \nabla T(V^*) \cdot f^* = \frac{\partial \left(\frac{\partial T}{\partial V_k^*} \right)}{\partial V} f(V) \quad (8-b)$$

According to the chosen canonical form, $\frac{\partial f^*}{\partial V_k^*}$ for $k = 1, \dots, n-1$ is

$$\frac{\partial f^*}{\partial V_k^*} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Namely, $(k+1)^{th}$ element is unity. Hence the second term on the right hand side is

$$\begin{cases} \frac{\partial T}{\partial V_{k+1}^*} & \text{for } 1 \leq k \leq n-1 \\ \frac{\partial T}{\partial V^*} \cdot \frac{\partial f^*}{\partial V_n^*} & \text{for } k = n \end{cases} \quad (8-c)$$

Therefore, from (8) using 8-a, b, c, for $1 \leq k \leq n-1$, the following is obtained:

$$\frac{\partial T}{\partial V_{k+1}^*} = \frac{\partial f}{\partial V} \frac{\partial T}{\partial V_k^*} - \frac{\partial \left(\frac{\partial T}{\partial V_k^*} \right)}{\partial V} f = \left[f, \frac{\partial T}{\partial V_k^*} \right] = \left[\text{ad}^1 f, \frac{\partial T}{\partial V_k^*} \right] \quad (9)$$

where $\left[f, \frac{\partial T}{\partial V_k^*} \right]$ is Lie bracket of the two vectors, f and $\frac{\partial T}{\partial V_k^*}$. From (3), if the measurement y is differentiated, then

$$\frac{\partial y}{\partial V^*} = \frac{\partial y}{\partial V} \cdot \frac{\partial V}{\partial V^*} = C \quad (10)$$

which can be reduced to

$$\langle \nabla h(V), \frac{\partial T}{\partial V^*} \rangle = C \quad (11)$$

Elements of the left hand side vector are

$$\langle \nabla h(V), \frac{\partial T}{\partial V_1^*} \rangle = 0 \quad (12)$$

$$\langle \nabla h(V), \frac{\partial T}{\partial V_2^*} \rangle = \langle \nabla h(V), \left[f, \frac{\partial T}{\partial V_1^*} \right] \rangle \quad (13)$$

From Leibnitz formula

$$\langle \nabla h(V), \left[f, \frac{\partial T}{\partial V_2^*} \right] \rangle = \langle \nabla \langle \nabla h, f \rangle, \frac{\partial T}{\partial V_1^*} \rangle - \langle \nabla \langle \nabla h, \frac{\partial T}{\partial V_1^*} \rangle, f \rangle \quad (14)$$

But considering the zero inner product in (12)

$$\langle \nabla h, \left[f, \frac{\partial T}{\partial V_1^*} \right] \rangle = \langle \nabla \langle \nabla h, f \rangle, \frac{\partial T}{\partial V_1^*} \rangle \quad (15)$$

where $\langle \nabla h, f \rangle \equiv L_f(h)$ is Lie derivative of h with respect to vector field f .

Therefore,

$$\langle \nabla h, \frac{\partial T}{\partial V_2^*} \rangle = \langle \nabla L_f(h), \frac{\partial T}{\partial V_1^*} \rangle \quad (16)$$

Similarly, it will be found that

$$\begin{aligned} \langle \nabla h, \frac{\partial T}{\partial V_3^*} \rangle &= \langle \nabla L_f^2(h), \frac{\partial T}{\partial V_1^*} \rangle \\ &\vdots \\ \langle \nabla h, \frac{\partial T}{\partial V_n^*} \rangle &= \langle \nabla L_f^{n-1}(h), \frac{\partial T}{\partial V_1^*} \rangle \end{aligned}$$

$$\frac{\partial h}{\partial V} \frac{\partial V}{\partial V^*} = C = \begin{bmatrix} \nabla h \\ \nabla L_f(h) \\ \nabla L_f^2(h) \\ \nabla L_f^3(h) \\ \vdots \\ \nabla L_f^{n-1}(h) \end{bmatrix} \frac{\partial T}{\partial V_1^*} \quad (17)$$

Lie derivatives of higher orders $L_f^2(h), L_f^3(h), \dots$, are defined as:

$$L_f^2(h) = L_f(L_f(h))$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$L_f^k(h) = L_f(\overbrace{L_f(\dots L_f(h))}^k) \dots$$

In order to have system of equations (1) transformable to observable form of equation (3) the transformation set T and its derivative must exist. Hence, $\frac{\partial T}{\partial V_i^*}$ must exist and in equation (16) the observability matrix O should be invertible.

$$O = \begin{bmatrix} \nabla h \\ \nabla L_f(h) \\ \vdots \\ \nabla L_f^{n-1}(h) \end{bmatrix}$$

This condition can be met by

$$\det O \neq 0 \quad (18)$$

From this point on two questions are considered. One being the condition for which the set of transformations T can be found, i.e., solvability of partial differential equation in terms of derivatives of T with respect to transformed state variables, i.e., V_i^* , $i=1, \dots, n$. The condition for solvability of this partial differential equation is given by Forbenius solvability condition, which requires the set of vectors

$$\frac{\partial T}{\partial V^*} = \left[\left[\text{ad}^0 f, \frac{\partial T}{\partial V_1^*} \right], \left[\text{ad}^1 f, \frac{\partial T}{\partial V_1^*} \right], \dots, \left[\text{ad}^{n-1} f, \frac{\partial T}{\partial V_1^*} \right] \right]$$

be involutive [33]. A set of vectors $\{f_i\}_{i=1, \dots, n}$ is involutive if the Lie bracket of any two is a linear combination of the $\{f_i\}$, i.e.

$$[f_i(V), f_j(V)] = \sum_{k=1}^n C_k f_k(V) \quad (19)$$

This condition is the tailored form of the fact that each transformed state can be solved in terms of state variable

$$V_i^* = V_i^*(V_1, \dots, V_n) \quad i=1, \dots, n \quad (20)$$

and the functional dependence is invertible.

Reconstruction of States from Measurements

Similar to equation (1) the system equation is

$$\begin{aligned} \dot{V} &= f(V, U) & V \in \mathcal{R}^n, & & U \in \mathcal{R}^m \\ y &= h(V) & y \in \mathcal{R} & \end{aligned} \quad (21)$$

Where y is the measurement scalar function, differentiating it successively with respect to time would result in:

$$\dot{y} = \frac{\partial h}{\partial t} = \frac{\partial h}{\partial V} \cdot \frac{\partial V}{\partial t}$$

Substitution for \dot{V} results in

$$\dot{y} = \frac{\partial h}{\partial V} \cdot f = \nabla h \cdot f$$

Similarly,

$$\begin{aligned} \ddot{y} &= \nabla(\nabla h \cdot f) \cdot \frac{\partial V}{\partial t} = \nabla L_f(h) \cdot f \\ &\vdots \\ y^{(n)} &= \nabla L_f^{n-1}(h) \cdot f \end{aligned} \quad (22)$$

Since y is a measurement, then all derivatives can be considered available. Therefore the problem of observability of the system would be reduced to the conditions under which all state variables are determined by the virtue of set of equations (22).

$$Y = \begin{bmatrix} \dot{y} \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} \nabla h \\ \nabla L_f(h) \\ \vdots \\ \nabla L_f^{n-1}(h) \end{bmatrix} \cdot f(V, U) \quad (23)$$

In order that vector function $f(V, U)$ be determined, again the inverse of observability matrix derived earlier for transformation to observable form must exist, namely

$$\det O = \det \begin{bmatrix} \nabla h \\ \nabla L_f(h) \\ \vdots \\ \nabla L_f^{n-1}(h) \end{bmatrix} \neq 0 \quad (24)$$

Hence

$$f(V, U) = [O]^{-1} \cdot Y \quad (25)$$

This condition is similar to the first condition for the existence of transformation from nonlinear form to canonical form. Based on (25), the state variables can be determined from the measurement vector Y if

$$\det \left[\frac{\partial f(V, U)}{\partial V} \right] \neq 0 \quad (26)$$

this condition, though not related to the existence of any transformation of system of (21) to observable form, guarantees the existence of at least one set of state variables as functions of measurements, and input(s).

$$V = V(Y, U)$$

Observability Analysis of MPD Engine

From conservation laws for species in plasma, a one-dimensional global model can be derived. In this paper the application of observability analysis to the global model is presented. However, similar steps can be taken to achieve more laborious task of determination of observability criteria for species state equations. Those global equations are:

$$\text{continuity: } \frac{\partial \rho}{\partial t} = - \left[\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right] = f_1 \quad (27-a)$$

$$\text{momentum: } \frac{\partial u}{\partial t} = - \left[u \frac{\partial u}{\partial x} + \frac{RT}{\rho} \frac{\partial \rho}{\partial x} + R \frac{\partial T}{\partial x} - \frac{\sigma B}{\rho} (E - uB) \right] = f_2 \quad (27-b)$$

$$\text{energy: } \frac{\partial T}{\partial t} = - \left[u \frac{\partial T}{\partial x} + \frac{RT}{C_v} \frac{\partial u}{\partial x} + \frac{K}{\rho C_v} \frac{\partial^2 T}{\partial x^2} - \frac{\sigma}{\rho C_v} E(E - uB) \right] = f_3 \quad (27-c)$$

It is assumed that shear stress and energy dissipation due to shear and collision along with radiation pressure and radiation energy are negligible. Also, in compliance with species set of equations and for simplicity of having one control variable, the applied magnetic field is assumed to be small and hence with negligible effects, i.e.

$$B \simeq 0$$

Hence the control variables are

$$U_1 = B[E - uB] \simeq 0, \quad U_2 = E[E - uB] = E^2 \quad (28)$$

Two cases for available measurements are considered. In the first case, density is assumed the only measurement, and in the second case temperature is chosen.

Case I: Density Measurement

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho \\ u \\ T \end{bmatrix} = \rho(x, t) \quad (29)$$

For this system the observability matrix can be determined from $\nabla h, \nabla L_f, \nabla L_f^2$.

$$\nabla h = \nabla C \cdot V = C = [1, 0, 0]$$

$$L_f = \nabla h \cdot f = - \left[\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right]$$

$$\nabla L_f = - \left\{ \left[u_1 + \frac{\rho_2}{\rho_1} u \right], \left[\rho_1 + \frac{\rho u_2}{u_1} \right], 0 \right\}$$

where index "i" is defined as an abbreviation for "i" times differentiation with respect to x.

$$L_f^2 = \nabla L_f \cdot f = \left[u_1 + \frac{\rho_2}{\rho_1} u \right] (u \rho_1 + \rho u_1) + \left[\rho_1 + \frac{\rho u_2}{u_1} \right] \left(u u_1 + \frac{RT}{\rho} \rho_1 + RT_1 \right)$$

$$\nabla L_f^2 = \left\{ K_1(\cdot), K_2(\cdot), \left[\rho_1 + \frac{\rho_2}{\rho_1} u \right] \left[\frac{R \rho_1}{\rho} + R \frac{T_2}{T_1} \right] \right\}$$

Hence,

$$O = \begin{bmatrix} 1 & 0 & 0 \\ - \left[u_1 + \frac{\rho_2}{\rho_1} u \right] & - \left[\rho_1 + \frac{\rho u_2}{u_1} \right] & 0 \\ K_1(\cdot) & K_2(\cdot) & \left[\rho_1 + \frac{\rho u_2}{u_1} \right] \left[\frac{R \rho_1}{\rho} + \frac{RT_2}{T_1} \right] \end{bmatrix} \quad (30)$$

The first condition for the observability of the system is

$$\det O \neq 0$$

This condition is satisfied if both of the following conditions are met:

$$\left[\rho_1 u_1 + \rho u_2 \right] = \frac{\partial}{\partial x} \left[\rho \frac{\partial u}{\partial x} \right] \neq 0 \quad (31)$$

and

$$\left[\rho_1 T_1 + \rho T_2 \right] = \frac{\partial}{\partial x} \left[\rho \frac{\partial T}{\partial x} \right] \neq 0 \quad (32)$$

The second condition given by (26) [and in a more restricted form by (19)] can be determined for the system. However the implicit function theorem which results in condition (26) for finite dimension system, will be modified for infinite dimensional system. In equation (25), the vector function in the left hand side (i.e. $f(V, U)$), for finite dimensional system would be a nonlinear algebraic vector function, whereas in the case of infinite dimensional system it will be a nonlinear ordinary differential equation. Therefore, the implicit function theorem will be considered in the space of 1-Jets [34]. The set of singular points of the equation:

$$F(x, V(x), \frac{dV}{dx}) = 0$$

with $P \equiv \frac{dV}{dx}$, in three dimensional space (x, V, P) of jets are those for which the tangent surface to $F(x, V, P) = 0$ is normal to (X, V) - plane, i.e. $\frac{\partial F}{\partial P} = 0$. Applying this criteria to the sets of equation (25) would result in the fact that the second condition will always be satisfied. This condition, states that for every input to the system there exists an equilibrium or null solution for the system, which is predictable for physical systems.

Consider the Venn diagrams in Figure (2) of set "I", satisfying the condition

$$\frac{\partial}{\partial x} \left[\rho \frac{\partial u}{\partial x} \right] \neq 0$$

and "II" satisfying the condition

$$\frac{\partial}{\partial x} \left[\rho \frac{\partial T}{\partial x} \right] \neq 0$$

where "S" is the set satisfying the system equations (27). If the cross section of sets

$$\frac{\partial}{\partial x} \left[\rho \frac{\partial u}{\partial x} \right] = \frac{\partial}{\partial x} \left[\rho \frac{\partial T}{\partial x} \right] = 0$$

with the system is found (i.e. solved), the resulting manifold, i.e. $S \cap \bar{I} \cap \bar{II}$ will represent unobservable manifold of S.

From equations (31) and (32), $S \cap \bar{I} \cap \bar{II}$ is determined by

$$\left. \begin{aligned} u_1 &= \frac{C_u(t)}{\rho}, \quad \phi(x,t) = \int_0^x \frac{dx}{\rho} \\ u(x,t) &= C_u \phi(x,t) + u_o(t) \\ T_1 &= \frac{C_T(t)}{\rho} \\ T(x,t) &= C_T(t) \phi(x,t) + T_o(t) \end{aligned} \right\} \quad (33)$$

where $u_o(t)$ and $T_o(t)$ are the given boundary conditions at $x = 0$ for velocity and temperature. The time functions $C_u(t)$ and $C_T(t)$ are constants of integration with respect to x . Solution of (33), by considering the MPD system equations, results in the following observability criteria:

$$U_2 \neq \frac{\left\{ \left[\dot{C}_T \phi + C_T \dot{\phi} + \dot{T}_o \right] + \left[(C_u \phi + u_o) \frac{C_T}{\rho} + \frac{R}{C_v} \frac{C_u}{\rho} (C_T \phi + T_o) - \frac{K}{\rho^3 C_v} \rho_1 C_T \right] \right\}}{\sigma / \rho^2 C_v} \quad (34)$$

However, U_2 is the control variable or input to the system. Therefore, to satisfy

observability criteria for MPD Engine, using density as the measurement, the control input should satisfy (34).

Case II: Temperature Measurement

If temperature is the available sensory information, then the measurement function of equation (29) will be modified to

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ u \\ T \end{bmatrix} \quad (35)$$

the same steps to construct the observability matrix are followed.

$$O = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{\rho^2 C_v} (\sigma U_2 - K T_2) & -\left(T_1 + \frac{RT}{C_v} \frac{u_2}{u_1}\right) & -\left(\frac{u T_2}{T_1} + u_1 \frac{R}{C_v} + \frac{K T_3}{\rho C_v T_1}\right) \\ K_p & K_u & K_T \end{bmatrix} \quad (36)$$

Rank condition requires that

$$\frac{K_u}{\rho^2 C_v} (\sigma u_2 - K T_2) - K_p \left(T_1 + \frac{RT}{C_v} \frac{u_2}{u_1}\right) \neq 0 \quad (37)$$

where K_p and K_u are notations used for very complex and lengthy terms.

Due to the complexity of the condition with this measurement the identification scheme with density measurement would result in faster and less complex process which makes density measurements more attractive.

Conclusion

The criteria for observability of finite and infinite dimensional system in terms of transformation of nonlinear systems to observable canonical form was studied. Similarly a less restrictive approach applicable to infinite dimensional systems, which results in general conditions for observability of such systems was presented. Application of this criteria to a global model for MPD system was developed and the region in time-spatial domain for control action was derived. Control action should avoid this region, so that the system behavior can be predicted from available measurements. Selection of measurements based on less complex criterion to satisfy was presented for the case of density measurements versus temperature measurements in an MPD engine.

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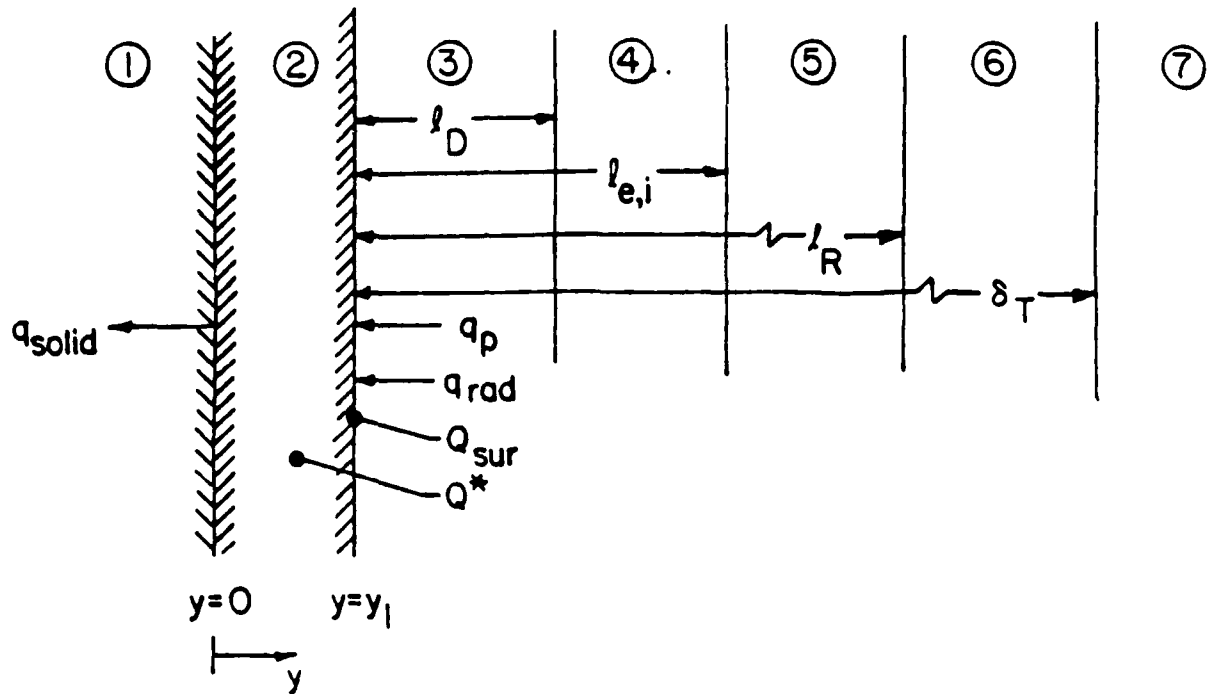


Figure 1.

Schematic of model for energy balance in the vicinity of a cathode of a MPD engine in the absence of an applied magnetic field. (1) solid conductor; (2) conductor in "mushy" state; (3) Debye region; (4) collisionless region outside sheath; (5) Region within a recombination length; (6) thermal boundary layer; (7) free stream.

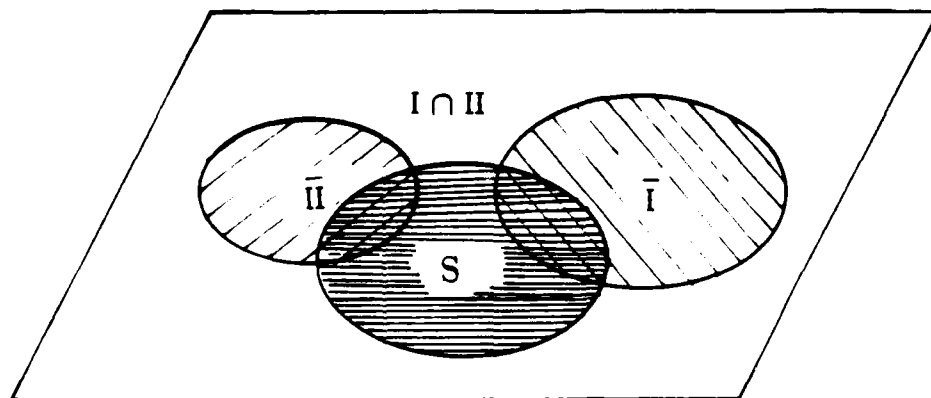


Figure 2.

Schematic Representation of Observability Region

NOMENCLATURE

Symbol	Description
\vec{B}	magnetic field
\vec{v}	particle velocity in laboratory reference frame
\vec{C}	peculiar particle velocity
\vec{E}	applied electric field
\vec{F}	body force on particle
\vec{J}	current flux
\vec{q}	heat flux
\vec{u}	mass velocity
\vec{U}	diffusion velocity in fluid reference frame
C_p	specific heat
e	electron charge
E	energy
h	Planck constant
k	Boltzmann constant
m	mass
n	number density
P	pressure
Q'	phase change energy
R	radiation energy
T	temperature
t	time
V	electric potential
W	potential energy

x position

Subscripts

cond conduction

e electron

i ion

k species of type k

mushy mushy region

n neutral

s species of type s

rad radiation

solid solid material region

sur surface

\parallel parallel to magnetic field

\perp perpendicular to magnetic field

H mutually perpendicular to magnetic and electric fields

α, β, γ directions

∞ free stream value

Greek

\mathcal{E} general electric field

ϵ_i ionization potential

η viscosity

λ, λ' thermal conductivity

ϕ thermal diffusion

ρ density

σ electrical conductivity

μ	mobility
$\bar{\nu}_{sk}$	mean collision frequency of type s particles with type k particles
τ	shear stress
q_s	particle charge
ℓ_D	Debye length
ℓ	mean free path

Definitions

$$\bar{b} = \frac{\bar{B}}{B}$$

$$\bar{C}_s = \text{mean particle speed} = \left\{ \frac{8kT_s}{\pi m_s} \right\}^{1/2}$$

$$\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$$

$$\vec{\tau} = \vec{E}' + \frac{\Delta P_e}{n_e e}$$

$f_s =$ velocity distribution function

$\rho_s = n_s m_s =$ species density

$$\rho = \sum_s \rho_s$$

$$m_{sk} = \text{reduced mass} = \frac{m_s m_k}{m_s + m_k}$$

$\dot{n}_s =$ generation rate of species s

$$\ell_R = \text{recombination length} = \left\{ \frac{2D_a}{\beta n_\infty^2} \right\}^{1/2}$$

$$D_a = \text{ambipolar diffusion coefficient} = \frac{\mu_e D_i + \mu_i D_e}{\mu_e + \mu_i}$$

$$D_i = \text{ion diffusion coefficient} = k \frac{T_h}{e} \mu_i$$

$$D_e = \text{electron diffusion coefficient} = k \frac{T_e}{e} \mu_e$$

$$\mu_e = \text{electron mobility} = \frac{e}{m_e \bar{v}_{eH}}$$

$$\mu_i = \text{ion mobility} = \frac{\rho_n}{\rho} \frac{\bar{v}_{en}}{\bar{v}_{eH}} + \frac{\rho_i}{\rho} \mu_{ie}$$

$$\mu_{in} = \frac{e}{m_{in} \bar{v}_{in}}$$

$$\mu_{ie} = \frac{e}{m_i \bar{v}_{eH}}$$

$$\bar{u}_s = \bar{u} + \bar{U}_s$$

$$\bar{u} = \frac{1}{\rho} \sum_s \rho_s \bar{U}_s$$

$$\ell = \text{mean free path} = \frac{\bar{c}}{\nu}$$

$$\frac{D(\)}{Dt} \equiv \frac{\partial(\)}{\partial t} + \bar{u} \cdot \nabla(\)$$

APPENDIX I

MODEL OF PLASMA IN THE VICINITY OF AN ELECTRODE

A partially ionized two-temperature non-equilibrium plasma flow is considered, which obeys Maxwell-Boltzmann statistics. The induced magnetic field is included for consideration.

The plasma is assumed to be composed of three monoatomic ideal gasses, electron, ion and neutral. The pressure of each gas species is given by $P_s = n_s k T_s$. The total gas pressure is then given by

$$P = \sum_s P_s \quad (I.1)$$

The energy of each species is the sum of the translational and the internal energies. The translational energy of each species is given by

$$e_s = \frac{3}{2} \frac{n_s k T_s}{m_s} \quad (I.2)$$

A Cartesian orthogonal coordinate system is used with coordinates that are parallel to the magnetic field, perpendicular to the magnetic field and perpendicular to the magnetic and the electric fields.

The electron current and heat flux are given by the following equations. The electron transport properties are significantly affected by the magnetic field and therefore these effects are included.

$$\begin{aligned} \vec{J}_e = & \sigma_{\parallel} \epsilon_{\parallel} + \sigma_{\perp} \epsilon_{\perp} + \sigma_H \vec{b} \times \epsilon + \phi_{\parallel} \nabla_{\parallel} T_e + \phi_{\perp} \nabla_{\perp} T_e \\ & + \phi_H \vec{b} \times \nabla T_e \end{aligned} \quad (I.3)$$

$$\begin{aligned} \vec{q}_e = & -\frac{5}{2} \frac{k T_e}{e} \vec{J}_e - \lambda'_{\parallel} \nabla_{\parallel} T_e - \lambda'_{\perp} \nabla_{\perp} T_e - \lambda'_H \vec{b} \times \nabla T_e \\ & - T_e \phi_{\parallel} \epsilon_{\parallel} - T_e \phi_{\perp} \epsilon_{\perp} - T_e \phi_H \vec{b} \times \epsilon \end{aligned} \quad (I.4)$$

The transport coefficients (σ , ϕ , and λ') are presented, for example in Mitchner and Kruger[I.1], in the form of integral functions of C . The transport coefficients are also available in other forms from sources such as Bose [I.2].

The heavy particle transport properties are not affected by the presence of a magnetic field unless the field is very strong. For the system considered the magnetic field is assumed to be sufficiently weak so that the heavy particle transport equations can be written for a partially ionized plasma without a magnetic field. Accordingly,

the following relations may be written for the heavy particle transport properties.

$$\tau_{\alpha\beta} = \eta \left\{ \frac{\partial u_{\alpha}}{\partial x_{\beta}} + \frac{\partial u_{\beta}}{\partial x_{\alpha}} - \frac{2}{3} (\nabla \cdot \mathbf{u}) \delta_{\alpha\beta} \right\} \quad (I.5)$$

$$\bar{q}_h = \frac{5}{2} k T_h \sum_s n_s \bar{U}_s - N \nabla T_h - n_h k T_h \sum_s \frac{D_s^T}{\rho_s} \bar{d}_s \quad (I.6)$$

$$\bar{J}_h = \bar{J}_i = e n_i \bar{U}_i \quad (I.7)$$

$$\bar{U}_s = \frac{n^2}{n_s \rho} \sum_k m_k D_{sk} \bar{d}_k - \frac{1}{\rho_s} D_s^T \nabla (\ln T_h) \quad (I.8)$$

$$\bar{d}_s \equiv \nabla \left(\frac{n_s}{n} \right) + \left(\frac{n_s}{n} - \frac{\rho_s}{\rho} \right) \nabla (\ln P) - \frac{\rho_s}{P \rho} \left(\frac{\rho}{m_s} \bar{F}_s - \sum_k n_k \bar{F}_k \right) \quad (I.9)$$

where

$\delta_{\alpha\beta}$ = Kronecker delta = 0 if $\alpha \neq \beta$, 1 if $\alpha = \beta$

D_{sk} = concentration diffusion coefficient

D_s^T = thermal diffusion coefficient

Since a non-equilibrium plasma is considered, a relationship for the generation of species is required. Only collisional reactions will be considered. Specifically the three-body recombination reaction where the third body is an electron may be written as follows.



The generation rate equation is then given by the following.

$$\dot{n}_e = \frac{\partial n_e}{\partial t} = \alpha(T_e) \left\{ (n_e n_i)^* - n_e n_i \right\} \quad (I.11)$$

where

$$(n_e n_i)^* \equiv n_n \left\{ \frac{n_e n_i}{n_n} \right\}_{\text{equil}} \quad (\text{I.12})$$

$$\alpha(T_e) = n_e 1.09 \times 10^{-20} T_e^{-9/2} \frac{m^3}{s} \quad (\text{I.13})$$

= Hinnoy-Hirschberg recombination coefficient

The equilibrium concentrations are given by the Saha equation, namely.

$$\left\{ \frac{n_e n_i}{n_n} \right\}_{\text{equil}} = 2 \frac{g_i}{g_n} \left\{ \frac{2\pi m_e k T_e}{h^2} \right\}^{\frac{3}{2}} \exp \left\{ \frac{-\epsilon_i}{k T_e} \right\} \quad (\text{I.14})$$

where g_s is equal to the electron energy partition function and ϵ_i , the ionizational energy. Since a singly ionized gas is considered, the electron and ion generation rates are equal ($\dot{n}_e = \dot{n}_i$).

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- I.1. Mitchner, M., Krugar, C.H., Partially Ionized Gases, John Wiley and Sons, Inc., 1974.
- I.2. Bose, T.K., Thermophysical and Transport Properties of Multi-Component Gas Plasmas at Multiple Temperatures, to be published in Progress in Aeronautical Sciences, 1987.

APPENDIX II

ENERGY BALANCE IN THE VICINITY OF A CATHODE

The energy equations are written specifically for each region shown in Figure 1. In all regions the plasma is assumed to be non-neutral ($n_e \neq n_i$) and two temperature governed ($T_e \neq T_i$).

REGION 1 (solid)

The energy equation in the solid material is given by the well-known Stephan or heat equation, namely.

$$\dot{E}_{st} = (\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} \quad (\text{II.1a})$$

or

$$\rho_{\text{solid}} C_{\text{solid}} \frac{\partial T}{\partial t} = \nabla(\lambda_{\text{solid}} \nabla T) + \frac{J^2}{\sigma_{\text{solid}}} \quad (\text{II.1b})$$

where \dot{E}_{st} is the energy stored in the material, $(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})$, net energy transfer through the material by conduction, and \dot{E}_{gen} , heat generated within the material due to Joule heating.

The current flux is given by

$$\vec{J} = \sigma_{\text{solid}} \vec{E} \quad (\text{II.2})$$

The energy input to the solid region is determined from an energy balance at the surface $y = 0$. The energy balance yields namely,

$$[\lambda_{\text{solid}} \frac{\partial T_{\text{solid}}}{\partial y}]_{y=0} = [\lambda_{\text{mushy}} \frac{\partial T_{\text{mushy}}}{\partial y}]_{y=0} + H_m \frac{\partial y}{\partial t} \quad (\text{II.3})$$

where H_m is the latent heat of fusion of the material

REGION 2 (Mushy)

Region 2 is the "mushy" region where solid, liquid and gaseous states may be present simultaneously. The energy for the "mushy" region is similar to the energy equation for the solid region except that an additional term is needed to account for the energy associated with the material phase change, $\dot{Q}'(T)$. If the material is assumed to "pure" then there will be no conduction across the mushy region. The region will be at a uniform temperature because it is undergoing a phase change. If the material is not considered "pure" then there can be conduction across the region because there may be a small temperature gradient. The energy transfer in the mushy region is then given by the following model equation (Reference II.1.), which is in the nature of a modified Stephan equation.

$$\rho_{\text{mushy}} C_{\text{mushy}} \frac{\partial T}{\partial t} = \nabla(\lambda_{\text{mushy}} \nabla T) + \frac{J^2}{\sigma_{\text{mushy}}} + \dot{Q}'(T) \quad (\text{II.4})$$

where

$$\vec{J} = \sigma_{\text{mushy}} \vec{E} \quad (\text{II.5})$$

and $\sigma_{\text{mushy}} = \sigma_{\text{mushy}}(T)$ = electrical conductivity of the mushy region which is temperature dependent

The two surfaces of the mushy region at $y = 0$ and $y = y_1$, are not fixed. The surface at $y = 0$ is allowed to recess into the solid material as melting occurs. Melting will occur in the region as long as the energy input to the region (at the surface $y = y_1$ and from Joule heating) exceeds the energy used in the phase transition plus the energy removed to the solid region by conduction. If the energy inputs and outputs are equal the boundary will not recede. The surface at $y = y_1$ will recede due to the evaporation of material at the surface.

The energy input to the mushy region from the plasma is determined from an energy balance evaluated at the surface $y = y_1$, as shown in Figure 1. The cathode is assumed to have a catalytic surface where incident ions and electrons recombine and are re-emitted as neutrals. Thermionic emission may also occur if the cathode temperature is sufficiently high. It should be noted that the thermionically emitted electrons provide an additional localized current which in turn produces an additional localized Joule heating. The localized heating may lead to localized evaporation or to the eruption of material. The overall energy balance equation is then as follows.

$$\dot{Q}^* + \dot{q}_{\text{cond}} = \dot{q}_p + \dot{q}_{\text{rad}} + \dot{Q}_{\text{sur}} \quad (\text{II.6})$$

where \dot{q}_p is the net energy transfer to the surface from the plasma due to the kinetic energy and potential energy of the particles, \dot{q}_{rad} , the net radiation to the surface from the plasma, \dot{Q}_{sur} , the energy generation at the surface due to recombination of ions and electrons \dot{Q}^* , the energy associated with phase change and, \dot{q}_{cond} , energy conducted into the mushy region.

REGION 3 (Sheath)

The region immediately adjacent to the electrode and within a distance of the order of Debye length (ℓ_D) of the surface is the sheath region. In this region, charge separation occurs and a net negative charge exists because of an excess number of electrons. The region is considered collisionless in the sense that only electron-neutral and ion-neutral collisions are present. These collisions are included because of the large number of neutrals in the region. Since neutrals being emitted from the surface do not experience a force from the fields, they tend to remain near the surface. They are removed through diffusion driven by the concentration gradient. All other collisions are assumed to be negligible.

The energy transfer to the surface from the plasma is from the impact of particles on the surface. Since a cathode is being considered the particles of interest are the ions. The energy of the particles is in two forms, the kinetic energy due to their motion and the potential energy associated with moving charged particles through an electric potential. The electric potential will tend to move the electrons away from the electrode while accelerating the ions towards the electrode. The neutral particles possess.

Since the interest is in particles which strike the surface, the velocity component normal to the surface is the one of interest. The describing energy equations are given by the following.

The species heat-flux vector, \vec{q}_s , is

$$\vec{q}_s = \int_{-\infty}^{\infty} \frac{1}{2} n_s m_s C^2 C_1 f_s d^3c \quad (II.7)$$

and the particle potential energy, W_s , is

$$W_s = \int_{y=y_1}^y \zeta_s V_s dy \quad (II.8)$$

where the electric potential V_s is given by the following Poisson's equation.

$$\nabla^2 V_s = \int_{-\infty}^{\infty} \zeta_s f_s d^3c \quad (II.9)$$

The species current-flux vector, \vec{J}_s , given by

$$\vec{J}_s = \int_{-\infty}^{\infty} n_s \zeta_s C_1 C_1 f_s d^3c \quad (II.10)$$

The distribution f_s is determined from the Boltzmann equation, namely

$$\begin{aligned} \frac{\partial}{\partial t} (n_s f_s) + (u_1 + C_1) \frac{\partial}{\partial x_1} (n_s f_s) + \left\{ \frac{F_{s1}}{m_s} - \frac{Du_1}{Dt} \right\} n_s \frac{\partial f_s}{\partial C} \\ - C_1 n_s \frac{\partial f_s}{\partial C_1} \frac{\partial u_s}{\partial x_1} = \sum_k \Psi_{sk} \end{aligned} \quad (II.11)$$

where Ψ_{sk} is the rate of increase of the property of interest (mass, momentum, energy or charge) due to collisions between particles of type s with particles of type k

REGION 4

Region 4 is very similar to region 3, as they are both considered collisionless. The describing equations are therefore the same for the two regions. Since this region is outside of the sheath, it is expected to contain a greater number of ions than region 3. The electron-neutral and ion-neutral collisions also become negligible because of a large decrease in the number of neutrals in the region. The collision parameter Ψ_{sk} in equation II.11 is therefore equal to zero.

REGIONS 5 and 6

Regions 5 and 6 are assumed to be collision-dominated and can therefore be described using the hydrodynamic approximation. Particle recombination is assumed to be absent in region 5 because it is within a distance of the order of the recombination length ℓ_R of the electrode surface. The recombination is included in region 6. The resulting energy balance equations for Region 5 are as follows.

$$\begin{aligned} \frac{D}{Dt} \left(\frac{3}{2} n_e k T_e \right) + \left(\frac{5}{2} n_e k T_e \right) \nabla \cdot \vec{u}_e &= -\nabla \cdot \vec{q}_e + \tau_e \nabla \vec{u}_e + \vec{J}_e \cdot \vec{E}'_e \\ &- \frac{2m_e}{m_h} \bar{\nu}_{eh} n_e \frac{3}{2} k (T_e - T_h) - \dot{R}_e \end{aligned} \quad (II.12)$$

$$\begin{aligned} \frac{D}{Dt} \left(\frac{3}{2} n_h k T_h \right) + \frac{5}{2} n_h k T_h \nabla \cdot \vec{u}_h &= -\nabla \cdot \vec{q}_h + \tau_h \nabla \vec{u}_h + \vec{J}_i \cdot \vec{E}'_i \\ &- \frac{2m_e}{m_h} \bar{\nu}_{he} n_h \frac{3}{2} k (T_h - T_e) - \dot{R}_h \end{aligned} \quad (II.13)$$

where

$$\tau \nabla \vec{u} = \tau \left(\frac{\partial \vec{u}}{\partial x} \right); \quad n_h = n_i + n_n; \quad m_n \approx m_i \approx m_h$$

The transport equations are given in appendix I.

The energy equations for region 6 are given by the following. The recombination energy is included in the electron energy equation.

$$\begin{aligned} \frac{D}{Dt} (n_e (\frac{3}{2} k T_e + \epsilon_i)) + n_e (\frac{5}{2} k T_e + \epsilon_i) \nabla \cdot \vec{u}_e &= -\nabla \cdot (\vec{q}_e - \frac{\epsilon_i}{e} \vec{J}_e) \\ &+ \tau_e \nabla \vec{u}_e + \vec{J}_e \cdot \vec{E}'_e - \frac{2m_e}{m_h} \bar{\nu}_{eh} n_e \frac{3}{2} k (T_e - T_h) \\ &- \dot{R}_e \end{aligned} \quad (II.14)$$

$$\begin{aligned} \frac{D}{Dt} \left(\frac{3}{2} n_h k T_h \right) + \frac{5}{2} n_h k T_h \nabla \cdot \vec{u}_h &= -\nabla \cdot \vec{q}_h + \tau_h \nabla \vec{u}_h + \vec{J}_i \cdot \vec{E}'_i \\ &- \frac{2m_e}{m_h} \bar{\nu}_{he} n_h \frac{3}{2} k (T_h - T_e) - \dot{R}_h \end{aligned} \quad (II.15)$$

REGION 7 (Free Stream)

In the free stream region, the various gradients are assumed to be very small compared to the other terms and are neglected. The rate of change of energy within the system is equal to the generated energy from Joule heating minus the energy lost from radiation, recombination and collisions. The energy equations can therefore be simplified as follows.

$$\frac{\partial}{\partial t} (n_e (\frac{3}{2} kT_e + \epsilon_i)) = \vec{J}_e \cdot \vec{E}'_e - \frac{2m_e}{m_h} \bar{v}_{eh} n_e \frac{3}{2} k(T_e - T_h) - \dot{R}_e \quad (\text{II.16})$$

$$\frac{\partial}{\partial t} (\frac{3}{2} n_h kT_h) = \vec{J}_i \cdot \vec{E}'_i - \frac{2m_e}{m_h} \bar{v}_{he} n_h \frac{3}{2} k(T_h - T_e) - \dot{R}_h \quad (\text{II.17})$$

$$\vec{J}_e = \sigma_{||} \epsilon_{||} + \sigma_{\perp} \epsilon_{\perp} + \sigma_H \vec{b} \times \epsilon \quad (\text{II.18})$$

REFERENCE

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CONTROLLABILITY ANALYSIS OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS*

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ABSTRACT

A method for analyzing controllability of nonlinear distributed parameter systems is presented. This analysis is based on the Lie algebra augmented for distributed systems. The augmentation is performed by transformation of the state equations into a finite dimensional state space. The resulting controllability criteria is applied to a nonlinear, distributed electric propulsion system, namely, a magneto-plasma dynamic (MPD) engine.

INTRODUCTION

Distributed parameter systems are often approximated or modelled by a lumped parameter approach or by discretization of the system in time and/or space [1,2]. This approach, although it has been useful for design of some control systems, in general cannot be applied to an accurate dynamic analysis of nonlinear distributed systems. This is especially true when considerations of general stability, controllability or observability of the system is concerned. Therefore, more accurate methods should be developed to investigate the general behavior of the nonlinear distributed systems. Regarding the questions of stability, many authors have studied the nonlinear distributed parameter systems. One of the first attempts using Lyapunov's direct method for infinite dimensional systems was made by Massera [3]. Based on an abstract theory of Lyapunov stability scheme of infinite dimensional system, references [4,5] have defined a Lyapunov function on Hilbert space and Banach space cases.

For finite dimensional systems the concept of controllability, was introduced in early 1960's by Kalman and others for linear systems, and in the early 1970's by work of Herman [6] and Haynes and Hermes [7] for nonlinear systems. The nonlinear analogy of linear controllability criteria for lumped parameter systems is carried out independently by Sussman-Jurdjevic [8] and Krener [9] by applying the Lie Algebra.

Controllability of a linear distributed parameter system has been investigated by researchers [10,11,12] for the cases that the control energy is either distributed in the system or is present at the boundary. As a recent attempt to derivation of an exact controllability criteria, Fattori and Russe [13] have used reduction of a special parabolic system to a moment problem and related the controllability to the absolute convergence of the moment problem.

In this paper a transformation is presented to augment the state space of nonlinear distributed parameter system. This transformation reduces the system into an equivalent set of finite dimensional nonlinear systems. The method proposed by [14,15] which transforms the finite dimensional nonlinear system into a canonical form is employed. Based on this transformation, a controllability criteria for nonlinear distributed parameter systems is established.

SYSTEM REPRESENTATION

Distributed parameter systems can be represented in many different forms. However, in terms of energetic systems where conservation laws and Maxwell equations may be applied, a distributed parameter system can be represented by the following form of state equations.

$$\frac{\partial V}{\partial t} = A(V) + B(V)U \quad (1)$$

where $V(x,t)$ is a state vector in a Banach space, which is a function of time (t) and space dimension (x), and it is composed of:

$$V(x,t) = \text{Col} \left[V_1(x,t), V_2(x,t), \dots, V_n(x,t) \right] \quad (2)$$

The operator A is a nonlinear operator defined on the Banach space. The specific case considered here is when A forms a general nonlinear function F^* of the following form:

$$A(V) = F^* \left[V, \frac{\partial V}{\partial x}, \dots, \frac{\partial^m V}{\partial x^m} \right] \quad (3)$$

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The input space consists of

$$U = \text{col} \left[U_1(x,t), U_2(x,t), \dots, U_r(x,t) \right] \quad (4)$$

Therefore, the operator $B(V)$ will be

$$B(V) = \left[B_1(V) \mid B_2(V) \mid \dots \mid B_r(V) \right] \quad (5)$$

with $B_i(V)$, $1 \leq i \leq r$, being a nonlinear operator of the form

$$B_i(V) = g_i \left[V, \frac{\partial V}{\partial x}, \dots, \frac{\partial^p V}{\partial x^p} \right] \quad (6)$$

with $p \leq m$. Equivalently, the general operator $B(V)$ can be represented as

$$B(V) = G^* \left[V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}, \dots, \frac{\partial^p V}{\partial x^p} \right] \quad (7)$$

where

$$G^* = \left[g_1^* \left[V, \dots, \frac{\partial^p V}{\partial x^p} \right] \mid g_2^* \left[\cdot \right] \mid \dots \mid g_r^* \left[\cdot \right] \right] \quad (8)$$

Equations (1), (3) and (8) represent a wide class of nonlinear distributed parameter systems. This system classification and representation is used in the following controllability derivations.

SYSTEM AUGMENTATION

In order to derive controllability criteria for the system of equation (1), the system is augmented such that the resulting equivalent system can be formulated in terms of a finite dimensional nonlinear system. This augmentation will reduce the infinite dimensional domains of the A and B operators into a finite domain at each point in the space dimension x .

Let

$$\frac{\partial V}{\partial x} = V^{(1)}, \quad \frac{\partial^2 V}{\partial x^2} = V^{(2)}, \dots \quad (9)$$

Therefore,

$$\begin{aligned} \frac{\partial V}{\partial t} &= F^* \left[V, V^{(1)}, V^{(2)}, \dots, V^{(m)} \right] \\ &+ G^* \left[V, V^{(1)}, V^{(2)}, \dots, V^{(p)} \right] U \end{aligned} \quad (10)$$

Differentiation with respect to x of the above state equation results in

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial V}{\partial t} \right) &= \frac{\partial}{\partial x} F^* \left[V, V^{(1)}, \dots, V^{(m)} \right] \\ &+ \frac{\partial}{\partial x} \left[G^* \left[V, V^{(1)}, \dots, V^{(p)} \right] U \right] \end{aligned} \quad (11)$$

The augmented state and input space are defined as

$$\Delta \begin{bmatrix} V \\ \frac{\partial V}{\partial x} \\ \frac{\partial^2 V}{\partial x^2} \\ \vdots \\ \frac{\partial^m V}{\partial x^m} \end{bmatrix} \quad (12) \quad \Delta \begin{bmatrix} U \\ \frac{\partial U}{\partial x} \\ \frac{\partial^2 U}{\partial x^2} \\ \vdots \\ \frac{\partial^m U}{\partial x^m} \end{bmatrix} \quad (13)$$

Assuming V , F^* , and G^* are analytical functions, then equation (11) can be written as:

$$\frac{\partial V^{(1)}}{\partial t} = \frac{\partial F^*}{\partial x}(\cdot) + \frac{\partial G^*}{\partial x} U + G^* \frac{\partial U}{\partial x}$$

Differentiating the state equation m times results in

$$\begin{aligned} \frac{\partial V^{(2)}}{\partial t} &= \frac{\partial^2 F^*}{\partial x^2}(\cdot) + \frac{\partial^2 G^*}{\partial x^2} U + \frac{2 \partial G^*}{\partial x} \frac{\partial U}{\partial x} + \frac{G^* \partial^2 U}{\partial x^2} \\ &\vdots \end{aligned} \quad (14)$$

$$\frac{\partial V^{(m)}}{\partial t} = \frac{\partial^m F^*}{\partial x^m}(\cdot) + \frac{\partial^m G^*}{\partial x^m} U + \dots + \frac{G^* \partial^m U}{\partial x^m}$$

Or in the augmented space the system of equation (11) can be written as

$$\frac{\partial \Delta}{\partial t} = F(\Delta) + G(\Delta)U \quad (15)$$

where

$$F(\Delta) = \begin{bmatrix} F^*(\Delta) \\ \frac{\partial}{\partial x} F^*(\Delta) \\ \vdots \\ \frac{\partial^m F^*}{\partial x^m}(\Delta) \end{bmatrix} \quad (16)$$

The augmented form for G^* , G , would be

$$G = \begin{bmatrix} G^* & 0 & 0 & 0 & 0 \\ \frac{\partial G^*}{\partial x} & G^* & 0 & 0 & 0 \\ \frac{\partial^2 G^*}{\partial x^2} & 2 \frac{\partial G^*}{\partial x} & G^* & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^m G^*}{\partial x^m} & \left[\begin{matrix} m \\ 1 \end{matrix} \right] \frac{\partial^{m-1} G^*}{\partial x^{m-1}} & \left[\begin{matrix} m \\ 2 \end{matrix} \right] \frac{\partial^{m-2} G^*}{\partial x^{m-2}} & \left[\begin{matrix} m \\ m-1 \end{matrix} \right] \frac{\partial G^*}{\partial x} & G^* \end{bmatrix}$$

Equation (15) is an augmented form of equation (1) and (10) which can be represented in an augmented finite dimensional space at each x with $F: M \rightarrow M$, and $G: M \rightarrow M$.

DERIVATION OF CONTROLLABILITY CRITERIA

In the previous section, it was shown that a nonlinear distributed system can be represented in an equivalent nonlinear finite dimensional system in the following form.

$$\dot{x}(t) = F(x) + G(x)u(t) \quad (18)$$

where $x \in R^n$, $u \in R^q$, $q = nm$

In general, controllability concept is related to relationship between inputs and states [6]. Namely, each state variable $x \in M$ can be manipulated by an input $u \in U$. In order to establish this relationship, the system equations should be transferred into a canonical form, as shown in the following.

$$\dot{x}^* = f(x^*) + g[u] \quad (19)$$

where

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_q^* \end{bmatrix}, \quad f(x^*) = \begin{bmatrix} f_1(x^*) \\ f_2(x^*) \\ \vdots \\ f_q(x^*) \end{bmatrix}, \quad \text{and} \quad g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The transformation that generates $x^* \in M^*$ from $x \in M$ is a one-to-one mapping $T(x): M \rightarrow M^*$. Hence

$$x^* = T(x) \quad x \in M \subseteq R^n \quad (20)$$

and inverse mapping is denoted by $\tilde{T}(x^*): M^* \rightarrow M$

$$x = \tilde{T}(x^*) \quad x^* \in M^* \subseteq R^n \quad (21)$$

Therefore,

$$\frac{d}{dt} (x^*) = \frac{d}{dt} [T(x)] = \frac{\partial T}{\partial x} \dot{x} = \frac{\partial T}{\partial x} [F + G u] \quad (22)$$

Comparison of equations (19), (20) and (22) results in

$$\frac{\partial T}{\partial x} F = f(g^*) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_q(x) \\ a(x) \end{bmatrix} \quad (23)$$

or

$$\frac{\partial T}{\partial x} F = T_{i+1} \quad \text{for } i = 1, \dots, q-1 \quad (24)$$

Since $T_i(x)$ is a scalar function, then

$$\frac{\partial T_i}{\partial x} F = \sum_{j=1}^q \frac{\partial T_i}{\partial x_j} F_j = \nabla T_i F = \langle dT_i, F \rangle \quad (25)$$

Therefore

$$T_{i+1} = \langle dT_i, F \rangle \quad \text{for } i = 1, \dots, q-1 \quad (26)$$

Hence

$$T_2 = \langle dT_1, F \rangle$$

$$T_3 = \langle d\langle dT_1, F \rangle, F \rangle$$

Furthermore, combination of equations (19) and (22) yields

$$\frac{\partial T}{\partial x} G(x) = g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

or equivalently

$$\langle dT_i, G(x) \rangle = 0, \quad i = 1, \dots, q-1 \quad (27)$$

Combination of (26) and (27) results in

$$\langle dT_{i+1}, G(x) \rangle = \langle d\langle dT_i, F \rangle, G \rangle$$

By direct expansion it can be proved that

$$\begin{aligned} \langle d\langle dT_i, F \rangle, G \rangle &= - \langle dT_i, \left[\frac{\partial G}{\partial x} F - \frac{\partial F}{\partial x} G \right] \rangle \\ &+ \langle d\langle dT_i, G \rangle, F \rangle \end{aligned}$$

The vector $\left[\frac{\partial T}{\partial x} F - \frac{\partial F}{\partial x} G \right]$ is a Lie bracket and can equivalently be represented by the Lie bracket F, G or $(\text{ad}_F G)$. Moreover, it was shown by (27) that the inner product $\langle dT_i, G \rangle$ is zero. Therefore,

$$\begin{aligned} \langle dT_{i+1}, G(x) \rangle &= \langle d\langle dT_i, F \rangle, G \rangle, \\ \langle dT_{i+1}, G(x) \rangle &= - \langle dT_i, \text{ad}_F^i G \rangle = 0 \end{aligned}$$

Similarly,

$$\langle dT_{i+1}, G(x) \rangle = \langle dT_i, \text{ad}_F^i G \rangle$$

where

$$\begin{aligned} \text{ad}_F^2 G &= [F, [F, G]] \\ \text{ad}_F^3 G &= [F, [F, [F, G]]] \end{aligned}$$

Since all of the inner product terms have the common vector $dT_1 = \frac{T_1}{\rho \lambda}$, then the above results can be summarized in a matrix notation as follows.

$$\frac{T_1}{\rho \lambda} \begin{bmatrix} G(\lambda) (ad^1 F, G) & (ad^2 F, G) & \cdots & (ad^{q-1} F, G) \end{bmatrix} = \begin{bmatrix} 0, 0, \dots, (-1)^{q-1} \end{bmatrix} \quad (28)$$

Let define

$$C = \begin{bmatrix} G(\lambda) (ad^1 F, G) & \cdots & (ad^{q-1} F, G) \end{bmatrix} \quad (29)$$

Then (28) can be written as

$$\frac{T_1}{\rho \lambda} = \begin{bmatrix} 0, 0, \dots, (-1)^{q-1} \end{bmatrix} C^{-1}$$

In order to have controllability for the system represented by (18) or (19), $\frac{T_1}{\rho \lambda}$ has to exist. This requires that the controllability matrix C has to be invertible. Therefore, controllability analysis of the nonlinear distributed parameter system would result in checking the rank of the C matrix shown by equation (29).

APPLICATION TO MPD ENGINE

1 - System Formulation

Recent increase in space missions and construction of the space station has attracted attention to new alternatives to chemical propulsion systems. One such system is an electric propulsion engine or magnetoplasma dynamic (MPD) engine [17].

Consider a simplified model for a Magneto Plasma Dynamic Thruster which has the applied magnetic field, B , and applied electric field, E , as the input control variables. The general governing conservation equations for the system can be written as

a) Continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0$$

For a one-dimensional model the above equation can be reduced to:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \quad (30)$$

where ρ and u are the plasma density and velocity, respectively.

b) The Momentum equation can be written in the x-direction as

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = - \frac{\partial P}{\partial x} + JB \quad (31)$$

It is assumed that $\frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0$, and the

electromagnetic force $\bar{J} \times \bar{B}$ is applied in x direction. J is the local current between the electrodes.

$$J = J(x, t)$$

c) One-dimensional non-steady energy equation will lead to:

$$\rho c_v \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right] = -P \frac{\partial u}{\partial x} + JE \quad (32)$$

In general, pressure P is a nonlinear function of density, ρ , and temperature, T , i.e. the state variables. In case of perfect gases, this relation is

$$P = P(\rho, T) = \rho RT \quad (33)$$

In the energy equation it is assumed that diffusion term is negligible compared to the heat generated inside the flow and the energy resulted from compressibility effects [18]. The applied fields are related to the current density by the Ohm's law;

$$J = \sigma(E - uB) \quad (34)$$

where the parameter σ is electrical conductivity and it is a property of the plasma.

Rewriting equations (30), (31) and (32) with respect to substitutions of equations (33) and (34) results in a set of state equations as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \quad (35a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{R}{\rho} \frac{\partial (\rho T)}{\partial x} = \frac{\sigma B(E - uB)}{\rho} \quad (35b)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + \frac{RT}{c_v} \frac{\partial u}{\partial x} = \frac{\sigma E(E - uB)}{\rho c_v} \quad (35c)$$

The set of equation (35) can be formulated in a distributed space form as:

$$\frac{\partial V}{\partial t} = F^* \left[V, \frac{\partial V}{\partial x} \right] + G^*(V)U \quad (36)$$

The control action U for this formulation will be

$$U = \begin{bmatrix} B(E - uB) \\ E(E - uB) \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

The distributed state variables are

$$V = \begin{bmatrix} \rho \\ u \\ T \end{bmatrix}$$

ii - Controllability Analysis

For the system of equations (35) the controllability technique presented in this paper is applied. In relation to the form of equation (36), it can be shown:

$$F^* \left[V, \frac{\partial V}{\partial x} \right] = - \begin{bmatrix} \frac{\partial}{\partial x} (\rho u) \\ u \frac{\partial u}{\partial x} + \frac{R}{\rho} \frac{\partial}{\partial x} (\rho T) \\ u \frac{\partial T}{\partial x} + \frac{RT}{c_v} \frac{\partial u}{\partial x} \end{bmatrix} \quad (37)$$

and

$$G^* = \begin{bmatrix} g_1(v) & g_2(v) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{\sigma}{\rho} & 0 \\ 0 & \frac{\sigma}{\rho c_v} \end{bmatrix} \quad (38)$$

Therefore, from (12), (13), (16) and (17) it can be shown that

$$\begin{bmatrix} u \\ \rho \\ T \\ \frac{\partial u}{\partial x} \\ \frac{\partial \rho}{\partial x} \\ \frac{\partial T}{\partial x} \end{bmatrix} = \begin{bmatrix} u \\ \rho \\ T \\ u^{(1)} \\ \rho^{(1)} \\ T^{(1)} \end{bmatrix}, \quad J = \begin{bmatrix} U_1 \\ U_2 \\ \frac{\partial U_1}{\partial x} \\ \frac{\partial U_2}{\partial x} \end{bmatrix}$$

$$F = - \begin{bmatrix} u\rho^{(1)} + \rho u^{(1)} \\ uu^{(1)} + \frac{RT}{\rho} \rho^{(1)} + RT^{(1)} \\ uT^{(1)} + \frac{RT}{c_v} u^{(1)} \\ \frac{\partial}{\partial x} \left[u\rho^{(1)} + \rho u^{(1)} \right] \\ \frac{\partial}{\partial x} \left[uu^{(1)} + \rho^{(1)} \frac{RT}{\rho} + RT^{(1)} \right] \\ \frac{\partial}{\partial x} \left[uT^{(1)} + \frac{RT}{c_v} u^{(1)} \right] \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\sigma}{\rho} & 0 & 0 & 0 \\ 0 & \frac{\sigma}{\rho c_v} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} \left[\frac{\sigma}{\rho} \right] & 0 & \frac{\sigma}{\rho} & 0 \\ 0 & \frac{\partial}{\partial x} \left[\frac{\sigma}{\rho c_v} \right] & 0 & \frac{\sigma}{\rho c_v} \end{bmatrix}$$

$$= \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \end{bmatrix}$$

Based on equation (29), the controllability matrix C can be constructed.

$$C = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & (ad^1 F, g_1) & (ad^1 F, g_2) & (ad^1 F, g_3) & (ad^1 F, g_4) \\ (ad^2 F, g_1) & (ad^2 F, g_2) & (ad^2 F, g_3) & (ad^2 F, g_4) \end{bmatrix} \quad (39)$$

The rank condition for this controllability matrix would lead to the criteria for the system controllability. As shown before, the terms in the controllability matrix (39) can be calculated as:

$$ad^1 F, g_1 = \left[F, g_1 \right] = \frac{\partial g_1}{\partial x} F - \frac{\partial F}{\partial x} g_1$$

$$\text{ad}^2 F \cdot g_1 = \left[F, \left[F, g_1 \right] \right] = \frac{(\text{ad}^1 F \cdot g_1)}{\partial \lambda} F - \frac{\partial F}{\partial \lambda} (\text{ad}^1 F \cdot g_1)$$

The same would be considered for g_1 , g_3 and g_4 in the calculation of ad^1 and ad^2 .

$$\frac{\partial F}{\partial \lambda} = - \begin{bmatrix} u^{(1)} & v^{(1)} & 0 & u & v & 0 \\ -\frac{\rho^{(1)}RT}{\rho^2} & T^{(1)} & \frac{R}{\rho} & \frac{RT}{\rho} & u & R \\ 0 & T^{(1)} & \frac{Ru^{(1)}}{C_v} & 0 & \frac{RT}{C_v} & u \\ \frac{u^{(1)}}{x} & \frac{v^{(1)}}{x} & 0 & \frac{u}{x} & \frac{v}{x} & 0 \\ -\frac{\rho^{(1)}RT}{\rho^2} & \frac{u^{(1)}}{x} & \frac{v^{(1)}}{x} & \frac{\rho^{(1)}R}{\rho} & \frac{RT}{\rho} & \frac{u}{x} \\ 0 & \frac{T^{(1)}}{x} & \frac{Ru^{(1)}}{C_v} & 0 & \frac{RT}{C_v} & \frac{u}{x} \end{bmatrix}$$

Following the detailed calculation to establish the matrix C, the rank condition of matrix C would be satisfied if and only if

$$\frac{\sigma^6}{\rho^6 c_v^2} \frac{\partial p}{\partial x} \frac{\partial}{\partial x} (\rho u) \neq 0 \quad (40)$$

Based on the fact that σ is not zero, both $\frac{\partial p}{\partial x} \neq 0$ and $\frac{\partial}{\partial x} (\rho u) \neq 0$ provide the controllability conditions for the operation of the MPD engine. Therefore, to have a controllable engine neither density nor momentum of the flow should be constant along the engine axes.

CONCLUSION

Controllability of nonlinear distributed parameter systems was studied. A criteria for controllability analysis was developed. This technique is based on applications of Lie algebra for augmented nonlinear distributed systems. The augmentation method, which involves two transformations of the distributed equations, was presented. The resulting controllability criteria was applied to a magneto-plasma dynamic (MPD) engine. This electric propulsion engine has nonlinear characteristics and is a distributed parameter system.

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Stability Analysis of Electro-Magneto-Plasma Dynamics*

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Abstract

An electro-magnetor-plasma process involves nonlinear, coupled, and complex dynamics and is subjected to a number of instability modes, some of them being associated with ionization or electro-thermal instability. Systems involved in such processes are more complicated due to the spatial and time variations in the system states (distributed parameter system). The objective of this paper is to present a new approach to stability of such distributed parameter systems. This technique is based on spatial extension of Lyapunov stability theorem. This Lyapunov functional approach is applied to a magneto-gas dynamic problem, and stability results are presented and discussed.

Introduction

Dynamics of systems and processes which involve multi-energy modes and interactions between fluid, thermal, and electromagnetic fields can generally be expressed by partial differential equations. Electric propulsion systems such as Magneto-Plasma-Dynamic (MPD) engines are examples of such systems. Stability analysis of such systems is very complicated by the fact that state variables are functions of both space and time. In this paper a new approach for stability analysis of such distributed parameter system is presented. This approach is based on the Lyapunov's direct stability method. Lyapunov stability theorem has become an important vehicle in derivation of stability analysis of solutions to linear and nonlinear ordinary differential equations. This approach attempts to make statements about stability of motions of a dynamical system without any knowledge of the solutions to its governing equations.

Although the development of Lyapunov's stability theorem for ordinary differential equations has been widely investigated, its application to solutions of partial differential equations (distributed parameter systems) has been limited. Most of the stability results for distributed parameter systems have been derived by use of approximation methods. These methods, in general, use reduction of the partial differential equations to a system of ordinary differential equations by spatial discretization or by assuming a harmonic time dependence. In the harmonic case the Galerkin method based on a truncation of the modal expansion is used. There have

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been studies in the applications of Lyapunov's stability theorem to distributed parameter systems (DPS) without any approximation. First attempt to apply Lyapunov's direct method for DPS was made by Massera [1] and Zubov [2]. Massera extended Lyapunov's method to denumerably infinite system of ordinary differential equations to arrive at a functional in terms of infinite dimensional state vector. This functional was used in place of the Lyapunov function. There have been studies in formulation of Lyapunov functions for special problems [3,4,5]. A general stability theorem based on the existence of a Lyapunov functional is established by Zubov in a functional space. Zubov has considered the general type of system, namely,

$$\frac{dZ}{dt}(t, \underline{X}) = L Z(t, \underline{X})$$

where $Z(t, \underline{X})$ is an n-dimensional vector valued function defined over some region Ω of spatial domain. Matrix valued operator L is a differential operation defined on the Ω .

The abstract theory of Lyapunov stability of infinite dimensional system was studied by Buis, Vogt, Eisen [6], Pau [7], and Banks [8]. In these studies Lyapunov function is defined by $\langle \underline{X}, S \underline{X} \rangle$ where S is a bounded positive self-adjoint operator, for Hilbert state space, and as $[\underline{X}, S \underline{X}]$ for the Banach space case, where $[\cdot, \cdot]$ is a semi-inner product.

Another approach to the stability analysis has been the application of the frequency domain methods. In this approach a generalized circle criterion is used for infinite dimensional systems [9,10,11].

Lyapunov Functional Approach

A distributed parameter system can be defined by the following equation

$$\frac{\partial \underline{X}}{\partial t} = \underline{f}(\underline{X}, \frac{\partial^n \underline{X}}{\partial \alpha^n}, \underline{U}, t) \quad (1)$$

where \underline{X} is the state vector, α is spatial coordinate, \underline{U} is the state input and t denotes time. If a Lyapunov functional $V: \mathcal{R}^n \rightarrow \mathcal{R}$ is defined for this system such that it satisfies the following

- i. $V(0) = 0$, and $V(\underline{X}) > 0$ for $\underline{X} \neq 0$ and $\underline{X} \in \mathcal{R}^n$
- ii. $V \in C^1(\mathcal{R}^n)$
- iii. $\dot{V} = \langle \text{grad} V, \underline{f} \rangle \leq 0$.

then the following two theorems can be stated [12].

Theorem 1: If in a neighborhood of the state origin there exists such a Lyapunov functional for the system (1), satisfying (i) through (iii), then the origin is a stable state for the system.

Theorem 2: If in a neighborhood of the origin there exists a Lyapunov functional V for the system (1), satisfying (i) through (iii) everywhere except at the origin itself, then the origin is an asymptotically stable state for the system.

Therefore, based on these theorems, stability of distributed parameter systems can be guaranteed provided a Lyapunov functional can be derived. In general, for systems defined in a Banach space, the functional V can be constructed based on a semi-inner product, namely,

$$V = [\underline{X}, P\underline{X}]$$

and for systems defined in a Hilbert space, the functional can be formulated by inner product, namely

$$V = \langle \underline{X}, P\underline{X} \rangle .$$

In the following section, derivation of such functional is shown for a simplified model of an MPD engine.

Problem Formulation:

Consider the MPD engine shown in Figure 1. The plasma dynamic equations for this engine consist of Maxwell's equations, Ohm's law, conservation of electric charge, equation of state (ideal gas law) and a set of mass, momentum and energy equations [24]. These equations can be written as follows:

Maxwell equations:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \mu \vec{H}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon} \rho_e$$

Ohm's law:

$$J_i = \sigma \left[E_i + \mu_e (\vec{U} \times \vec{H})_i \right] + \rho_e U_i \quad i = x, y, z$$

Conservation of electric charge:

$$\frac{\partial \rho_e}{\partial t} + \sum_{j=1}^3 \frac{\partial J_j}{\partial x_j} = 0$$

Equation of state (ideal gas law):

$$P = R\rho T$$

Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^3 \frac{\partial}{\partial x_j} (\rho U_j) = 0$$

Conservation of Momentum:

$$\rho \frac{DU_i}{Dt} = - \frac{\partial P}{\partial x_i} + \sum_{j=1}^3 \frac{\partial \tau_{ij}}{\partial x_j} + F_{e_i} + F_{g_i}$$

where,

$$F_{e_i} = \rho_e E_i + \mu_e (\vec{J} \times \vec{H})_i \quad i \triangleq x, y, z \text{ or } x_1, x_2, x_3$$

F_{g_i} = gravity force per volume

$$\tau_{ij} = \mu \frac{\partial U_i}{\partial x_j}$$

Energy Equation:

$$\frac{\partial \rho \bar{e}_m}{\partial t} + \sum_{j=1}^3 \frac{\partial \rho \bar{e}_m U_j}{\partial x_j} = - \frac{\partial U_j P}{\partial x_j} + \frac{\partial U_j \tau_{ij}}{\partial x_j} + E_j J_j + \frac{\partial Q_j}{\partial x_j}$$

where \bar{e}_m = total energy per unit mass = $c_v T$.

In this model it is assumed that the plasma is originally at rest with pressure P_0 , temperature T_0 , and density ρ_0 . An external uniform magnetic field H_0 is applied to the system where,

$$\vec{H}_0 = \vec{i} H_x + \vec{j} H_y + \vec{k} 0.$$

There is no electric field applied to the system. Plasma is perturbed by a small disturbance and as a result the state of the system is a combination of stationary (equilibrium) part and perturbed portion. For velocity vector the basic flow is zero. Therefore,

$$U = i u(x,t) + j v(x,t) + k w(x,t)$$

However, an assumption is made that the variations of variables are only functions of one spatial dimension, x , and time. Therefore, instantaneous pressure, temperature and density can be written as

$$\begin{aligned} P &= P_0 + P'(x,t) \\ T &= T_0 + T'(x,t) \\ \rho &= \rho_0 + \rho'(x,t) \end{aligned}$$

Electric and magnetic fields can be represented as

$$\begin{aligned} \vec{E} &= i E_x(x,t) + j E_y(x,t) + k E_z(x,t) \\ \vec{H} &= \vec{H}_0 + \vec{h}(x,t) \\ &= i \left[H_x + h_x(x,t) \right] + j \left[H_y + h_y(x,t) \right] + k h_z(x,t) \end{aligned}$$

current density J and net electric charge ρ_e are

$$\vec{J} = \vec{J}(x,t), \quad \rho_e = \rho_e(x,t).$$

The one-dimensional assumption results in: $\frac{\partial(\cdot)}{\partial y} = 0, \frac{\partial(\cdot)}{\partial z} = 0,$

Based on the aforementioned treatment of the problem, the following describing equations can be derived.

Maxwell's equations:

$$J_x + \epsilon \frac{\partial E_x}{\partial t} = 0 \quad (2)$$

$$J_y + \epsilon \frac{\partial E_y}{\partial t} = - \frac{\partial h_z}{\partial x} \quad (3)$$

$$J_z + \epsilon \frac{\partial E_z}{\partial t} = \frac{\partial h_y}{\partial x} \quad (4)$$

$$\mu_e \frac{\partial h_x}{\partial t} = 0 \quad (5)$$

$$\mu_e \frac{\partial h_y}{\partial t} = \frac{\partial E_z}{\partial x} \quad (6)$$

$$\mu_e \frac{\partial h_z}{\partial t} = - \frac{\partial E_y}{\partial x} \quad (7)$$

Generalized Ohm's law:

$$J_x = \sigma (E_x - \mu_e \omega H_y) + \rho_e u \quad (8)$$

$$J_y = \sigma (E_y + \mu_e w H_x) + \rho_e v \quad (9)$$

$$J_z = \sigma (E_z + \mu_e u H_y - \mu_e v H_x) + \rho_e w \quad (10)$$

Conservation of electric charge:

$$\frac{\partial \rho_e}{\partial t} + \frac{\partial J_x}{\partial x} = 0 \quad (11)$$

Equation of state for perturbed variables is

$$\frac{P'}{P_o} = \frac{\rho'}{\rho_o} + \frac{T'}{T_o}, \quad \text{where } P_o = \rho_o R T_o \quad (12)$$

Linearized continuity equation becomes:

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u}{\partial x} = 0 \quad (13)$$

Linearized equations of momentum are:

$$\rho_o \frac{\partial u}{\partial t} = - \frac{\partial P'}{\partial x} + \frac{4}{3} \mu \frac{\partial^2 u}{\partial x^2} - \mu_e J_z H_y + \rho_e E_x \quad (14)$$

$$\rho_o \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} + \mu_e J_z H_x + \rho_e E_y \quad (15)$$

$$\rho_o \frac{\partial w}{\partial t} = \mu \frac{\partial^2 w}{\partial x^2} + \mu_e (J_x H_y - H_x J_y) + \rho_e E_z \quad (16)$$

It is assumed that the nonlinear perturbation terms are negligible in comparison with the linear terms. Therefore, the energy equation becomes

$$\rho_o c_v \frac{\partial T'}{\partial t} = - \rho_o R T_o \frac{\partial u}{\partial x} + K \frac{\partial^2 T'}{\partial x^2} \quad (17)$$

Decoupled Modes of Motion

In case of a neutral plasma, i.e., $\rho_e \simeq 0$, the number of ions and electrons per volume of plasma are nearly equal. For this case, if one considers the fact that $\frac{\partial h_x}{\partial t} = 0$, $\frac{\partial h_x}{\partial x} = 0$, then it is possible to distinguish between two modes of wave propagation, transverse mode (z-direction) and longitudinal mode. In the transverse mode the states are found to be h_z and w , and the state equations can be formed from reduction of equations (2),(3),(7),(8),(9),(11), and (16). The rest of the equations can be reduced to form state equations for the longitudinal mode.

(i.) Transverse Mode

The magneto-gas-dynamic assumption results in insignificant magnetic induction effect in Maxwell's equations from the terms carrying variations of electric field with time. This is due to the fact that nondimensional parameters $R_t = \frac{t_0 U}{L}$ and $R_E = \frac{E_0}{\mu_e U H_0}$ are of the order of one or smaller, and $R_c = \frac{U^2}{c^2} = U^2 \mu_e \epsilon \ll 1$ [12]. The resulting equations are

$$\frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} + H_x \frac{\partial w}{\partial x} \quad (18)$$

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial x^2} + \frac{V_x^2}{H_x} \frac{\partial h_z}{\partial x} \quad (19)$$

where $\nu_H = \frac{1}{\sigma \mu_e}$ and $V_x = \sqrt{\frac{\mu_e}{\rho_0}} H_x$. The parameter V_x is defined as x-component of speed of Alfven wave.

(ii.) Longitudinal Mode

The state equation for this mode can be reduced to

$$\frac{\partial h_y}{\partial t} = \nu_H \frac{\partial^2 h_y}{\partial x^2} - H_y \frac{\partial u}{\partial x} + H_x \frac{\partial v}{\partial x} \quad (20)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} + \frac{V_x^2}{H_x} \frac{\partial h_y}{\partial x} \quad (21)$$

$$\frac{\partial u}{\partial t} = -RT_0 \left[\frac{\partial \rho''}{\partial x} + \frac{\partial T''}{\partial x} \right] + \frac{4}{3} \nu \frac{\partial^2 u}{\partial x^2} - \frac{V_y^2}{H_y} \frac{\partial h_y}{\partial x} \quad (22)$$

$$\frac{\partial \rho''}{\partial t} = - \frac{\partial u}{\partial x} \quad (23)$$

$$\frac{\partial T''}{\partial t} = \frac{K}{\rho_0 c_v} \frac{\partial^2 T''}{\partial x^2} - \frac{R}{c_v} \frac{\partial u}{\partial x} \quad (24)$$

where $\rho'' = \frac{\rho'}{\rho_0}$, $T'' = \frac{T'}{T_0}$, $V_y = \sqrt{\frac{\mu_e}{\rho_0}} H_y$. The parameter V_y is defined as

y-component of the speed of Alfvén wave.

Lyapunov Functional and Stability Analysis

In this section the Lyapunov Functional approach is applied to each mode of the plasma dynamics. The stability results are derived and discussed.

- (i.) For the transverse mode the general operator form of the equation (18) and (19) evolution equation would be

$$\frac{d\underline{Z}}{dt} = \underline{A}(\underline{Z}) \quad (25)$$

where

$$\underline{Z} = \begin{bmatrix} h_z \\ w \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} \nu_H \frac{\partial^2}{\partial x^2} & H_x \frac{\partial}{\partial x} \\ \frac{V_x^2}{H_x} \frac{\partial}{\partial x} & \nu \frac{\partial^2}{\partial x^2} \end{bmatrix}$$

$$\underline{Z} = \underline{Z}(x, t)$$

Assuming that the boundary conditions are

$$\underline{Z}(0, t) = 0, \quad \underline{Z}(\ell, t) = 0$$

The operator \underline{A} is defined in $L^2(0, \ell)$ and its domain belongs to a Hilbert space. A Lyapunov functional $V(\underline{Z}(x, t), t)$ should be constructed such that $V(\underline{Z}, t) > 0$ and $\frac{dV}{dt} < 0$. For this problem the functional V is chosen to be the inner product of \underline{Z} .

$$V = \langle \underline{Z}, \underline{Z} \rangle_1 = \langle \underline{Z}, \underline{PZ} \rangle_0, \quad \underline{P} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \quad \alpha_1, \alpha_2 > 0.$$

$$V = \alpha_1 \|h_z\|^2 + \alpha_2 \|w\|^2. \quad \text{Hence; } \dot{V} = \frac{d}{dt} \langle \underline{Z}, \underline{Z} \rangle_1 = 2 \langle \dot{\underline{Z}}, \underline{Z} \rangle_1 = 2 \langle \underline{AZ}, \underline{Z} \rangle_1.$$

Since \underline{P} is symmetrix $\langle x, \underline{Py} \rangle = \langle \underline{Px}, y \rangle$

$$\dot{V} = 2 \langle \underline{AZ}, \underline{PZ} \rangle_0 = 2 \langle \underline{PAZ}, \underline{Z} \rangle_0$$

where,

$$\langle \underline{PAZ}, \underline{Z} \rangle_0 = \int_0^\ell \underline{Z}^T \cdot \begin{bmatrix} \alpha_1 \nu_H \frac{\partial^2}{\partial x^2} & \alpha_1 H_x \frac{\partial}{\partial x} \\ \alpha_2 \frac{V_x^2}{H_x} \frac{\partial}{\partial x} & \alpha_2 \nu \frac{\partial^2}{\partial x^2} \end{bmatrix} \underline{Z} \, dx \quad (26)$$

Considering all of the terms in the above integral, the following can be resulted.

$$\dot{V} = 2 \int_0^{\ell} \left\{ h_z \left[\alpha_1 \nu_H \frac{\partial^2 h_z}{\partial x^2} + \alpha_1 H_x \frac{\partial w}{\partial x} \right] + w \left[\frac{\alpha_2 V_x^2}{H_x} \frac{\partial h_z}{\partial x} + \alpha_2 \nu \frac{\partial^2 w}{\partial x^2} \right] \right\} dx \quad (27)$$

For this problem since the eigenfunctions of states are similar, i.e., the Hilbert spaces of states have the same sequence of coordinates, if α_1 and α_2 are selected such that $\alpha_1/\alpha_2 = V_x^2/H_x^2$ then (27) can be reduced to

$$\dot{V} = 2 \int_0^{\ell} \left[\alpha_1 \nu_H h_z \frac{\partial^2 h_z}{\partial x^2} + \alpha_2 \nu w \frac{\partial^2 w}{\partial x^2} \right] dx + 2 \alpha_1 H_x \int_0^{\ell} \left[h_z \frac{\partial w}{\partial x} + w \frac{\partial h_z}{\partial x} \right] dx \quad (28)$$

Using integral by parts and the following relationships [13],

$$\int_0^{\ell} \left[\left(\frac{\partial f}{\partial x} \right)^2 \right] dx \geq \pi^2 \int_0^{\ell} f^2 dx$$

and

$$\int_0^{\ell} \left[h_z \frac{\partial w}{\partial x} + w \frac{\partial h_z}{\partial x} \right] dx = h_z w \Big|_0^{\ell} = 0$$

the following can be resulted.

$$\begin{aligned} \dot{V} &= -2 \int_0^{\ell} \left[\alpha_1 \nu_H \left(\frac{\partial h_z}{\partial x} \right)^2 + \alpha_2 \nu \left(\frac{\partial w}{\partial x} \right)^2 \right] dx \\ &\leq -2\pi^2 \int_0^{\ell} (\alpha_1 \nu_H h_z^2 + \alpha_2 \nu w^2) dx \end{aligned}$$

if $\alpha_2 = 1$, $\alpha_1 = V_x^2/H_x^2$ then

$$V = \frac{V_x^2}{H_x^2} \|h_z\|^2 + \|w\|^2$$

$$\dot{V} \leq -2\pi^2 \left[\frac{V_x^2}{H_x^2} \nu_H \|h_z\| + \nu \|w\|^2 \right] \quad (29)$$

In this case (29) shows that \dot{V} is negative definite and that proves the stability of the transverse mode using Lyapunov approach.

(ii.) For longitudinal mode of wave propagation the generic form of evolution equation (25) is considered

$$\underline{Z} = \begin{bmatrix} h_y \\ v \\ u \\ \rho'' \\ T'' \end{bmatrix} \in L^2(0, \ell) \quad \text{and } \underline{Z}(0, t) = \underline{Z}(\ell, t) = 0.$$

In this case \underline{A} in equation (25) is a linear operator with the domain in a separable Hilbert space of state function which maps \underline{Z} onto itself. From equations (20) to (24) \underline{A} is formed as,

$$\underline{A} = \begin{bmatrix} \nu_H \frac{\partial^2}{\partial x^2} & H_x \frac{\partial}{\partial x} & -H_y \frac{\partial}{\partial x} & 0 & 0 \\ \frac{V_x^2}{H_x} \frac{\partial}{\partial x} & \nu \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ -\frac{V_y^2}{H_y} \frac{\partial}{\partial x} & 0 & \frac{4}{3} \nu \frac{\partial^2}{\partial x^2} & -RT_0 \frac{\partial}{\partial x} & -RT_0 \frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{R}{C_v} \frac{\partial}{\partial x} & 0 & \frac{K}{\rho_0 C_v} \frac{\partial^2}{\partial x^2} \end{bmatrix}$$

An approach similar to the transverse mode is taken to construct the Lyapunov functional for the longitudinal mode.

$$V = \langle Z, PZ \rangle$$

where

$$P = \begin{bmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \alpha_3 & & \\ & & & \alpha_4 & \\ & & & & \alpha_5 \end{bmatrix}, \alpha_i > 0$$

Introduction of this form for V results in integral terms in \dot{V} in the following form

$$\int_0^\ell Z_i \frac{\partial Z_j}{\partial x} dx. \quad (30)$$

Such terms would make \dot{V} indefinite unless solution forms are assumed for the state

variables. In order to make the stability results independent of the solution forms, the following equalities are assumed.

$$\begin{aligned}\alpha_1 H_x &= \alpha_2 \frac{V_x^2}{H_x} \\ -\alpha_1 H_y &= -\alpha_3 \frac{V_y^2}{H_y} \\ -RT_o \alpha_3 &= -\alpha_4 \\ -RT_o \alpha_3 &= -\alpha_5 \frac{R}{C_v}\end{aligned}$$

These result in values for α_2 through α_4 and force all of the integral terms in form of (30) to go to zero. Assume α_1 is unity the following are resulted.

$$\begin{aligned}\alpha_1 &\triangleq 1 \\ \alpha_2 &= \frac{H_x^2}{V_x^2} \\ \alpha_3 &= \frac{H_y^2}{V_y^2} \\ \alpha_4 &= \frac{H_y^2}{V_y^2} RT \\ \alpha_5 &= \frac{H_y^2}{V_y^2} C_v T\end{aligned}$$

Based on these values for α_i 's and similar algebraic techniques used in the transverse mode, the following can be derived.

$$V = \sum_{i=1}^5 \alpha_i \|Z_i\|^2 > 0$$

and

$$\dot{V} \leq -2\pi^2 \left[\alpha_1 \nu_H \|h_y\|^2 + \alpha_2 \nu \|v\|^2 + \alpha_3 \frac{4}{3} \nu \|u\|^2 + \alpha_4 \frac{K}{\rho_o C_v} \|T''\|^2 \right]$$

These results indicate that for Z_i & $\alpha_i \neq 0$ V is positive and \dot{V} is negative. Therefore, the longitudinal mode is stable. Hence, stability of the MPD engine near its equilibrium is established without solving any of its dynamic equations.

Stability results only for the transverse mode of the MPD engine can be established by application of point spectrum approach. This would provide a means of checking the results presented in this paper, namely, by means of the Lyapunov functional

approach.

By means of separation of variables the semi-group property $T(t)$, generated by A can be constructed.

$$Z = \begin{bmatrix} T_b X_b \\ T_w X_w \end{bmatrix}$$

or

$$\begin{bmatrix} \frac{\dot{T}_b}{T_b} - \nu_H \frac{X_b''}{X_b} & H_x \frac{X_w'}{X_w} \\ \frac{V_x^2}{H_x} \frac{X_b'}{X_b} & \frac{\dot{T}_w}{T_w} - \nu \frac{X_w''}{X_w} \end{bmatrix} \begin{bmatrix} T_b X_b \\ T_w X_w \end{bmatrix} = 0$$

The solution for $Z = \begin{bmatrix} T_b X_b \\ T_w X_w \end{bmatrix}$ will exist if the above operation on Z is not one-to-one; i.e.,

$$\det \begin{bmatrix} \frac{\dot{T}_b}{T_b} - \nu_H \frac{X_b''}{X_b} & H_x \frac{X_w'}{X_w} \\ \frac{V_x^2}{H_x} \frac{X_b'}{X_b} & \frac{\dot{T}_w}{T_w} - \nu \frac{X_w''}{X_w} \end{bmatrix} = 0 \quad (31)$$

In order to have T and X functions independent from x and t , respectively, it is required that $\frac{\dot{T}}{T} \neq f(x,t)$, $\frac{X''}{X} \neq f(x,t)$. From $\frac{\dot{T}}{T}$, $\frac{X'}{X} = \text{const.}$ and $\frac{X''}{X} = \text{const.}$, it is conclusive to represent X functions in terms of a real periodic function

$$X = A_x e^{i\lambda x/\ell} + A_x^* e^{-i\lambda x/\ell}$$

where $A^* = \text{complex conjugate of } A$

$$\frac{X''}{X} = -\frac{\lambda^2}{\ell^2}, \quad \frac{X'}{X} = \frac{i\lambda}{\ell}$$

and $T = A_T e^{st} \rightarrow \frac{\dot{T}}{T} = S$. From the boundary condition it appears

$$X = A_x \sin \frac{\lambda_n}{\ell}$$

$$\lambda_n = n\pi \quad n = \pm 1, \pm 2, \dots$$

Therefore, equation (31) can be reduced to

$$\det \begin{bmatrix} S + \nu_H \frac{\lambda_n^2}{\ell^2} & -H_x i \frac{\lambda_n}{\ell} \\ -\frac{V_x^2}{H_x} i \frac{\lambda_n}{\ell} & S + \nu \frac{\lambda_n^2}{\ell^2} \end{bmatrix} = 0$$

The point spectrum of operator A, i.e., $\sigma_p(A)$ can be found as $\sigma_p(A) = \{S \in \sigma(A) \mid (SI-A) \text{ is not one to one}\}$

$$S^2 + S(\nu_H + \nu) \frac{\lambda_n^2}{\ell^2} + V_x^2 \frac{\lambda_n^2}{\ell^2} + \nu V_H \frac{\lambda_n^4}{\ell^4} = 0$$

The roots of this equation are

$$S_{n,1,2} = \frac{1}{2} \left[-(\nu_H + \nu) \frac{\lambda_n^2}{\ell^2} \pm \sqrt{(\nu_H - \nu)^2 \frac{\lambda_n^4}{\ell^4} - 4V_x^2 \frac{\lambda_n^2}{\ell^2}} \right]$$

The above expression for S_n indicates that $\sup [\operatorname{Re} \sigma(A)] < 0$ which is the necessary and sufficient condition for the equilibrium solution of equations (18) and (19) to be exponentially stable, i.e. an equivalence to uniform asymptotic stability for the above linear system.

The point spectrum approach applied to the longitudinal mode results in the following spectrum of the set of state equations.

$$\begin{aligned} & \left\{ K \left(\frac{1}{\rho_o} + \frac{4}{3} \frac{\nu}{p_o} S \right) \lambda^4 + \left[S^2 \frac{k}{p_o} + \frac{4}{3} \frac{\nu S^2}{T_o(\gamma-1)} + S C_p \right] \lambda^2 \right. \\ & \quad \left. + \frac{S^3}{T_o(\gamma-1)} \right\} \cdot \left[(\nu_H \lambda^2 + S)(\nu \lambda^2 + S) + V_x^2 \lambda^2 \right] \\ & \quad + \lambda^2 V_y^2 (S + \nu \lambda^2) \left[\frac{S^2}{T_o(\gamma-1)i} + \frac{S K \lambda^2}{p_o} \right] = 0 \end{aligned}$$

This characteristic equation doesn't have a closed form solution. Therefore, in general the spectrum approach doesn't yield stability solution for a system, unless in very simplified and special cases. Whereas the outlined Lyapunov approach would result in stability solutions for a system.

Conclusion

A new approach to stability analysis of an MPD engine was presented. This technique is based on the Lyapunov stability analysis which is extended to cover distributed parameter systems. The results of the stability analysis was supported with those derived from spectral analysis point of view. The procedure for construction of the Lyapunov functional and derivation of its derivative was presented. It was shown that while spectral technique can be applied to stability analysis of a special class of systems, the presented method can be applied to any form of distributed parameter systems.

Nomenclature

C_v :	Constant volume specific heat
C_p :	Const. press. specific heat
\bar{e}_m :	Internal energy/mass
\vec{E} :	Electric field vector
\vec{H}, h :	Magnetic field vector, perturbation vector of magnetic field.
\vec{H}_0 :	Constant magnetic field or characteristics magnetic field
\vec{J} :	Electric current density
K :	Thermal conductivity of plasma
L, ℓ :	Characteristic length, length in the x direction (for the flow)
P, p' :	Pressure, perturbation in pressure
Q :	Heat diffusion/area
R :	Gas constant
T, T' :	Temperature, perturbation in temperature
t, t_0 :	time, characteristic time
\vec{U}, U :	Velocity vector, characteristic velocity
u, v, w :	velocity vector components in x,y and z directions
x, y, z :	Spatial coordinates
X_j :	Spatial coordinates for $j = 1, 2, 3$

Greeks

ϵ :	Dielectric constant
μ, ν	Viscosity, kinematic viscosity
μ_e :	Magnetic permeability
ρ, ρ' :	Plasma density, perturbation in density

ρ_e :	Excess electric charge/volume
σ :	Electrical conductivity
ν_H :	$\frac{1}{\sigma\mu_e}$

subscripts

x, y, z: Variable subscribed in the direction of x, y and z.

Notation

T_h, T_w : Time dependent part of variable in subscript

X_h, X_w : x dependent part of variable in subscript.

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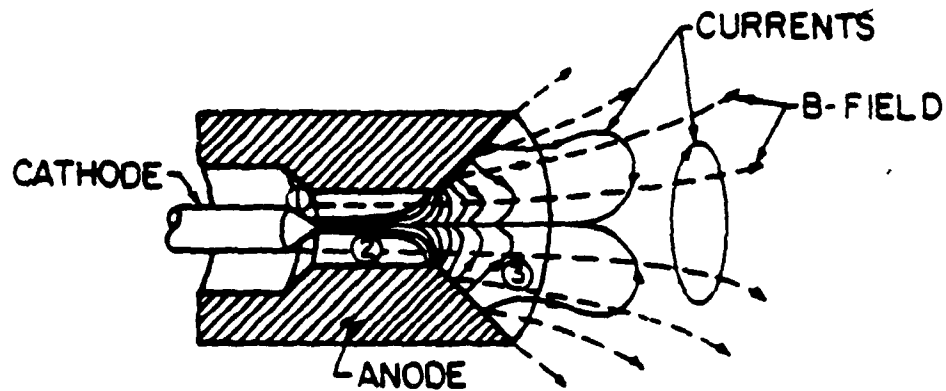


FIGURE 1. MPD ARC THRUSTER

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